

# THE CLASSIFICATION PROBLEM FOR FINITELY GENERATED OPERATOR SYSTEMS AND SPACES

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**ABSTRACT.** The classification of separable operator systems and spaces is commonly believed to be intractable. We analyze this belief from the point of view of Borel complexity theory. On one hand we confirm that the classification problems for arbitrary separable operator systems and spaces are intractable. On the other hand we show that the finitely generated operator systems and spaces are completely classifiable (or smooth); in fact a finitely generated operator system is classified by its complete theory when regarded as a structure in continuous logic. In the particular case of operator systems generated by a single unitary, a complete invariant is given by the spectrum of the unitary up to a rigid motion of the circle, provided that the spectrum contains at least 5 points. As a consequence of these results we show that the relation on compact subsets of  $\mathbb{C}^n$ , given by homeomorphism via a degree 1 map, is smooth.

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## 1. INTRODUCTION

An operator system is a closed unital self-adjoint subspace of a unital  $C^*$ -algebra. The study of operator systems can be traced back to the pioneering work of Arveson in the late 1960s and early 1970s [1, 2]. The foundational result of the theory is the Choi–Effros abstract characterization in terms of positive cones on matrix amplifications [15]; see also [36, Theorem 13.1]. Since then the theory of operator systems has expanded substantially thanks to the work of Blecher, Paulsen, and many others [36].

Operator systems offer a natural framework to study complete positivity, which is an essential tool in operator algebras, as well as a number of problems in quantum information theory. For instance the Tsirelson problem, teleportation, bounded entanglement, and superdense coding all admit equivalent reformulations in terms of operator systems [11–13, 31, 32]. Moreover Farenick, Kavruk, Paulsen, and Todorov found in [23, 24, 33] reformulations of the Connes embedding problem in terms of tensor products of finite-dimensional operator systems. Of course, the Connes embedding problem is one of the most important open problems in operator algebras.

In the last few years several great advances have occurred in the theory of operator systems. By work of Dritschell–McCullough, Arveson, Davidson–Kennedy and others [3, 16, 17] the existence of sufficiently many boundary representations to generate the  $C^*$ -envelope has been established, confirming Arveson’s intuition from almost 50 years earlier. (The existence of the  $C^*$ -envelope had been previously established by Hamana via his theory of injective envelopes [28, 29].) Despite these recent advances, classification results for operator systems are very rare. The only more or less explicit classification result to this date is Arveson’s classification of operator systems acting on a finite-dimensional Hilbert space from [4].

We recall the definition of unital completely positive map, which is the natural notion of morphism between operator systems. If  $\phi: X \rightarrow Y$  is a linear map between operator systems, then the  $n^{\text{th}}$  amplification  $\text{id}_{M_n(\mathbb{C})} \otimes \phi$  of  $\phi$  is the linear map from  $M_n(\mathbb{C}) \otimes X$  to  $M_n(\mathbb{C}) \otimes Y$  given by

$$\sum_i \alpha_i \otimes x_i \mapsto \sum_i \alpha_i \otimes \phi(x_i).$$

(Here,  $\otimes$  denotes the algebraic tensor product.) Under the canonical identification of  $M_n(\mathbb{C}) \otimes X$  and  $M_n(\mathbb{C}) \otimes Y$  with  $M_n(X)$  and  $M_n(Y)$  this map has the form

$$[x_{ij}] \mapsto [\phi(x_{ij})]$$

and it is often denoted by  $\phi^{(n)}$ . A linear map  $\phi: X \rightarrow Y$  is *unital completely positive* if  $\phi(1) = 1$  and for every  $n$  the amplification  $\phi^{(n)} = \text{id}_{M_n(\mathbb{C})} \otimes \phi$  is positive, i.e., it sends positive elements in  $M_n(X)$  to positive elements in  $M_n(Y)$ .

The operator systems together with the unital completely positive maps form a category, and isomorphism in this category is called *complete order*

*isomorphism*. It is worth recalling that a unital map between operator systems is completely positive if and only if it is completely contractive [36, Proposition 3.6]. As a consequence, complete order isomorphisms are precisely the unital invertible complete isometries. Moreover a complete order isomorphism between  $C^*$ -algebras is automatically multiplicative [8, Corollary 1.3.10], and hence a  $*$ -isomorphism. In the following, the classification of operator systems will always mean classification up to complete order isomorphism.

It seems to be commonly believed that any meaningful classification of even finitely generated operator systems is out of reach. In this paper we show that, while on one hand the classification problem for all separable operator systems is intractable due to a result of Sabok [40], on the other hand the classification problem for finitely generated operator systems is tractable. More precisely we prove that from the point of view of Borel complexity theory, the classification of finitely generated operator systems is as low in complexity as it can conceivably be.

Borel complexity theory is an area of mathematics that studies the relative complexity of classification problems using tools and methods of descriptive set theory. In this framework our notions of complexity and classifiability are made precise. Moreover powerful tools and criteria—such as Hjorth’s theory of turbulence [30]—have been developed to rigorously exclude the possibility of certain classification results.

In the setting of Borel complexity theory, a classification problem is regarded as an equivalence relation on a standard Borel space. This covers, perhaps after a suitable parametrization, most concrete classification problems in mathematics. The fundamental notion of comparison between equivalence relations is the following: if  $E, F$  are equivalence relations on standard Borel spaces  $X, Y$ , then  $E$  is *Borel reducible* to  $F$  if there exists a Borel function  $f: X \rightarrow Y$  satisfying

$$x E x' \iff f(x) F f(x').$$

Borel complexity theory provides a number of *benchmarks* of complexity to obtain a hierarchy of classification problems in mathematics. For example an equivalence relation is *smooth* or *concretely classifiable* if it is Borel reducible to the relation of equality on the set of real numbers (or any Polish space). Classification results of this type are the most satisfactory. For instance the structure theorem for countable divisible abelian groups shows that relation of isomorphism for such groups is smooth. Similarly the relation of isomorphism of UHF  $C^*$ -algebras is smooth [26]. But such results are quite strong and therefore rare. Indeed the classification of torsion-free abelian groups of rank 1 is already nonsmooth.

It is therefore natural to extend the notion of classifiability and to allow invariants that are higher in the complexity hierarchy. Consider the relation  $\cong_{\mathcal{C}}$  of isomorphism within some class  $\mathcal{C}$  of countable structures. A relation is *classifiable by countable structures* if it is Borel reducible to  $\cong_{\mathcal{C}}$

for some class of countable structures  $\mathcal{C}$ . Most concrete classification results in mathematics can be viewed as a classification by countable structures. For instance, in Elliott's celebrated classification of separable AF algebras from [18], the invariants include countable ordered abelian groups with a distinguished element (the ordered  $K_0$  groups).

Although classifiability by countable structures is a very general notion, there are many very complex classification problems that are not classifiable by countable structures. For examples, Sasyk–Törnquist and Farah–Toms–Törnquist have shown in [22, 41] using Hjorth's theory of turbulence that separable  $\text{II}_1$  factors and separable simple nuclear  $C^*$ -algebras are *not classifiable by countable structures*.

The next natural benchmark of complexity is given by orbit equivalence relations of Polish group actions. A relation is then *classifiable by orbits* if it is Borel reducible to the orbit equivalence relation associated with the continuous action of a Polish group on a Polish space. It was shown in [19] that separable operator systems are classifiable by orbits. In fact a result of Sabok from [40] shows that separable operator systems (and in fact even simple, separable, nuclear  $C^*$ -algebras) have *maximal complexity* among classes that are classifiable by orbits.

In this paper we show that the situation is very different for finitely generated operator systems. Namely the classification problem for finitely generated operator systems is smooth. We prove this result by showing that the complete order isomorphism classes of finitely generated operator systems are naturally parametrized by the points of a Polish space.

We give a second proof of the same fact using the framework of logic for metric structures. The logic for metric structures, or continuous logic, is a generalization of the usual first order logic that is suitable for application to functional-analytic structures such as operator systems and  $C^*$ -algebras. In this framework operator systems are regarded as structures in a suitable language  $\mathcal{L}_{OSy}$  [27, Appendix B]; see also [19, Section 3.3]. One can then assign to every operator system its theory as an  $\mathcal{L}_{OSy}$ -structure, which is the collection of evaluations of  $\mathcal{L}_{OSy}$ -formulas. For finitely generated operator systems this provides a concrete—albeit hard to calculate—smooth complete invariant. The same result applies to all collections of structures in some language for which the domain of quantification is compact, such as finite-dimensional operator spaces up to complete isometry.

In the even more special case of operator systems generated by a unitary we provide a more explicit complete invariant. For unitaries with three or less points in their spectra, isomorphism of the operator systems they generate is simply determined by the cardinality of the spectrum. In the case of unitaries with five or more points in their spectrum, two operator systems generated by unitaries are complete order isomorphic if and only if the spectra of the generating unitaries are conjugate by a rigid motion of the circle.

As a consequence of the smoothness result for finitely generated operator systems, we draw similar conclusions for a natural relation of degree 1 homeomorphism for compact subsets of  $\mathbb{C}$  or, more generally,  $\mathbb{C}^n$ . Here, compact subsets of  $\mathbb{C}^n$  are said to be *degree 1 homeomorphic* if they are homeomorphic via a linear combination of  $1$ ,  $z$ ,  $\bar{z}$ , and  $\bar{z}z$  (we call this a *degree 1 map*). For comparison, note that the classification of compact subsets of  $\mathbb{C}$  up to arbitrary homeomorphism is not classifiable by countable structures. This latter result is due to Farah–Toms–Törnquist and was obtained using the methods of [30].

We also consider the classification problem for separable (unital) operator spaces. An operator space is a linear subspace of a  $C^*$ -algebra, while a unital operator space is a unital linear subspace of a unital  $C^*$ -algebra. Similarly to operator systems, (unital) operator spaces admit abstract characterizations; see [39, Theorem 3.1] and [9, Theorem 1.1]. There are two natural relations of equivalence for (unital) operator spaces: (unital) complete isometry and (unital) completely bounded isomorphism. We show that the relation of complete isomorphism of separable operator spaces has maximal complexity among analytic equivalence relations. This follows directly from the analogous result for Banach spaces [25]. Moreover the relations of (unital) complete isometry of finitely generated (unital) operator spaces is smooth. (It should be noted that all  $N$ -dimensional operator spaces are completely isomorphic by [38, Corollary 7.7].)

In addition to establishing the results described above, in the present paper we take the broader aim of laying a foundation for the study of the complexity of classification problems for operator systems and spaces. (The initial results in this direction were obtained in [19].) To this purpose we consider many natural parametrizations of operator spaces and systems, and then we show that they are all (weakly) equivalent in the sense of [22]. (The analogous results for parametrizations of  $C^*$ -algebras have been obtained in [22].) As a consequence this implies that any of these natural parametrizations can be used to assess the complexity of some class of operator systems or spaces without affecting the conclusions.

The rest of the paper is organized as follows. In Section 2 we introduce many natural parametrizations of operator systems and spaces. The full proof of the equivalence of these parametrizations is given in the Appendices A and B. In Section 3 we consider the classification problem for arbitrary separable operator systems and spaces. In Section 4 we specialize the analysis to finitely generated operator systems, and show that they are concretely classifiable. Finally in Section 5 we present and prove the more general smoothness result for  $\mathcal{L}$ -structures in continuous logic.

In the following all the structures (Banach spaces, operator spaces, operator systems, and  $\mathcal{L}$ -structures) are assumed to be separable, complete, and nonzero. As usual in model theory we denote tuples of elements by  $\bar{x}$  and  $\bar{y}$ . We reserve the letter  $z$  for a complex variable, in which case  $\bar{z}$  will denote the complex conjugate of  $z$ . If  $X$  is a set and  $n \in \mathbb{N}$  we denote by

$M_n(X)$  the set of  $n \times n$  matrices with entries from  $X$ . If  $K$  is a field and  $X$  is a  $K$ -vector space, then  $M_n(X)$  will be identified with the  $K$ -vector space  $M_n(K) \otimes X$ .

The authors would like to thank David Blecher for referring them to the articles [9, 10].

## 2. PARAMETRIZING OPERATOR SYSTEMS AND OPERATOR SPACES

We consider in this section several natural standard Borel parametrizations of the categories **OSy** and **OSp** of complete separable operator systems and spaces. In Appendix A and Appendix B we will show that all these parametrizations are weakly equivalent.

**2.1. Standard Borel parametrizations.** Following [22, Definition 2.1] a *standard Borel parametrization* of a category  $\mathcal{C}$  is a pair  $(X, f)$  where  $X$  is a standard Borel space and  $f$  is a map from  $X$  to the class of objects of  $\mathcal{C}$ , such that the range of  $f$  contains an isomorphic copy of every object of  $\mathcal{C}$ . Two parametrizations  $(X, f)$  and  $(Y, g)$  are called *weakly equivalent* [22, Definition 2.1] if there are Borel functions  $a : X \rightarrow Y$  and  $b : Y \rightarrow X$  such that  $(g \circ a)(x) \cong f(x)$  and  $(f \circ b)(y) \cong g(y)$  for every  $x \in X$  and  $y \in Y$ . If moreover one can choose  $a$  and  $b$  to be injective, then the parametrizations  $(X, f)$  and  $(Y, g)$  are called *equivalent*. As observed in [22, Section 2] it follows from the Borel version of the Schröder–Bernstein theorem [34] that the parametrizations  $(X, f)$  and  $(Y, g)$  are equivalent if and only if there is a Borel isomorphism  $\varphi$  from  $X$  to  $Y$  such that  $(g \circ \varphi)(x) \cong f(x)$  and  $(f \circ \varphi^{-1})(y) \cong g(y)$  for every  $x \in X$  and  $y \in Y$ .

Suppose that  $\mathcal{C}$  is a category, and  $(X, f)$  is a parametrization of  $\mathcal{C}$ . A subcategory  $\mathcal{C}_0$  of  $\mathcal{C}$  is *Borel* (in the parametrization  $f$ ) if the set

$$X_0 = \{x \in X : f(x) \text{ is an object of } \mathcal{C}_0\}$$

is a Borel subset of  $X$ . The relation of isomorphism of  $\mathcal{C}_0$  in the parametrization  $f$  is the relation  $E_{\cong}^{\mathcal{C}_0}$  on  $X_0$  defined by

$$x E_{\cong}^{\mathcal{C}_0} x' \Leftrightarrow f(x) \cong f(x').$$

It is clear that replacing  $(X, f)$  with a weakly equivalent parametrization  $(X', f')$  does not change the notion of Borel subcategory. Moreover the isomorphism relation corresponding to the parametrization  $(X', f')$  is Borel bireducible with the isomorphism relation corresponding to the parametrization  $(X, f)$ .

**2.2. Parametrizations of operator systems.** A  $\mathbb{Q}(i)$ -\*-vector space  $S$  is a  $\mathbb{Q}(i)$ -vector space endowed with a conjugate linear involution  $x \mapsto x^*$ . Denote by  $\mathcal{V}$  the (countable)  $\mathbb{Q}(i)$ -\*-vector space of \*-polynomials of degree at most 1, with constant term, with coefficients in  $\mathbb{Q}(i)$ , in the noncommutative variables  $X_n$  for  $n \in \mathbb{N}$ . Similarly define  $\mathcal{V}_{\mathbb{C}}$  the complex \*-vector space of \*-polynomials of degree at most 1, with constant term, with complex coefficients, in the noncommutative variables  $X_j$  for  $j \in \mathbb{N}$ .

Suppose that  $\{\mathbf{p}_n : n \in \mathbb{N}\}$  is an enumeration of  $\mathcal{V}$  such that  $\mathbf{p}_1$  is the constant polynomial 1.

**2.2.1. The space  $\Gamma$ .** Let  $H$  be the separable infinite-dimensional Hilbert space. Denote by  $B(H)$  the algebra of bounded linear operators on  $H$ . For every  $n \in \mathbb{N}$  endow the  $n$ -ball  $B_n(H)$  of  $B(H)$  with the (compact metrizable) weak operator topology. Finally endow  $B(H)$  with the corresponding inductive limit (standard) Borel structure, obtained by setting  $A \subset B(H)$  Borel iff  $A \cap B_n(H)$  is Borel for every  $n \in \mathbb{N}$ . Denote by  $\Gamma$  the set  $B(H)^{\mathbb{N}}$  of sequences in  $B(H)$  endowed with the product Borel structure. This can be seen as a standard Borel parametrization of **OSy**. For  $\gamma \in \Gamma$  and  $p \in \mathcal{V}_{\mathbb{C}}$  define  $p(\gamma)$  to be the element of  $B(H)$  obtained by replacing the variable  $X_i$  with  $\gamma_i$  and interpreting a constant  $c$  as the corresponding multiple  $cI$  of the identity operator. Every element  $\gamma = (\gamma_n)_{n \in \mathbb{N}}$  of  $\Gamma$  codes the separable operator system  $\mathcal{OSy}(\gamma)$  obtained by taking the closure in the norm topology of the set  $\{p(\gamma) : p \in \mathcal{V}_{\mathbb{C}}\}$ .

The space  $\Gamma_{\mathcal{V}}$  of unital linear self-adjoint functions from  $\mathcal{V}$  to  $B(H)$  is a Borel subset of  $B(H)^{\mathcal{V}}$  endowed with the product Borel structure. This is also a standard Borel parametrization of **OSy**. An element  $\varphi$  of  $\Gamma_{\mathcal{V}}$  codes the operator system  $\mathcal{OSy}(\varphi)$  which is the closure of the range of  $\varphi$ . The function  $\gamma \mapsto \varphi_{\gamma}$  from  $\Gamma$  to  $\Gamma_{\mathcal{V}}$  defined by  $\varphi_{\gamma}(p) = p(\gamma)$  is a Borel isomorphism witnessing that the parametrizations  $\Gamma$  to  $\Gamma_{\mathcal{V}}$  are equivalent.

**2.2.2. The space  $\Xi$ .** Denote by  $\Xi$  the space of  $\delta = (\delta_n)_{n \in \mathbb{N}}$  where  $\delta_n \in \mathbb{R}^{M_n(\mathcal{V})}$  is such that for some operator system  $X$  and some nonzero dense sequence  $\gamma \in X$ ,

$$\delta_n([p_{ij}]) = \|[p_{ij}(\gamma)]\|_{M_n(X)}.$$

The ultraproduct construction shows that  $\Xi$  is a Borel set. The operator system  $\mathcal{OSy}(\delta)$  associated with  $\delta$ , which is completely isometric to  $X$  as above, can be described as the Hausdorff completion of  $\mathcal{V}$  with respect to the seminorm  $p \mapsto \delta_1(p)$ .

**2.2.3. The space  $\widehat{\Xi}$ .** Denote by  $M_n(S)$  the  $\mathbb{Q}(i)$ -\*-vector space of  $n \times n$  matrices over  $S$ . A *matrix order* on  $S$  is a collection  $(C_n)_{n \in \mathbb{N}}$  of cones  $C_n$  on  $M_n(S)$  such that

- (1)  $C_n \cap (-C_n) = \{0\}$ , and
- (2) for every  $n, m \in \mathbb{N}$  and every  $n \times m$  matrix  $A$  with coefficients in  $\mathbb{Q}(i)$ ,  $A^*C_nA \subset C_m$ .

We call a selfadjoint  $e \in S$  an *order unit* if for every selfadjoint  $x \in S$  there is  $r \in \mathbb{Q}_+$  such that  $re + x \in C_1$ . An order unit is *Archimedean* if  $re + x \in C_1$  for all  $r \in \mathbb{Q}_+$  implies  $x \in C_1$ . We call  $e$  an Archimedean matrix order unit provided that  $I_n \otimes e \in M_n(S)$  is an Archimedean order unit for  $M_n(S)$ .

Suppose  $S$  is a matrix ordered  $\mathbb{Q}(i)$ -\*-vector space with an Archimedean matrix order unit. The same argument as [36, page 176] shows that  $C_n$  is

a full cone for every  $n \in \mathbb{N}$ , i.e.  $C_n - C_n = M_n(S)$ . Moreover the proof of [36, Proposition 13.3] yields that

$$\|x\| = \inf \left\{ r : \begin{bmatrix} rI_n & x \\ x & rI_n \end{bmatrix} \in C_{2n} \right\}$$

is a norm on  $M_n(S)$ , and  $C_n$  is a closed subset of  $M_n(S)$  in the topology induced by such norm.

The completion  $\widehat{S}$  of  $S$  with respect to such norm is then a *complex*  $*$ -vector space. Moreover the closure  $\widehat{C}_n$  of  $C_n$  inside  $M_n(\widehat{S})$  for  $n \in \mathbb{N}$  form a matrix order on  $\widehat{S}$  with Archimedean matrix order unit  $e$ . Therefore by the abstract characterization of operator systems due to Choi and Effros [36, Theorem 13.1]  $S$  is completely isometrically isomorphic to an operator system.

In view of the above observations we consider the Borel space  $\widehat{\Xi}$  of tuples

$$\xi = (f_\xi, g_\xi, h_\xi, (C_{\xi,n})_{n \in \mathbb{N}}, e_\xi) \in \mathbb{N}^{\mathbb{N} \times \mathbb{N}} \times \mathbb{N}^{\mathbb{Q}(i) \times \mathbb{N}} \times \mathbb{N}^{\mathbb{N}} \times \prod_{n \in \mathbb{N}} 2^{\mathbb{N}^2} \times \mathbb{N}$$

that code on  $\mathbb{N}$  a  $\mathbb{Q}(i)$ - $*$ -vector space structure  $S_\xi$  by setting

$$\begin{aligned} n +_\xi m &= f_\xi(n, m), \\ q \cdot_\xi n &= g_\xi(q, n), \\ n^{*\xi} &= h_\xi(n), \end{aligned}$$

where  $C_{\xi,n} \subset M_n(\mathbb{N})$  is the positive cone, and  $e_\xi$  is the Archimedean matrix order unit. The corresponding norm on  $M_n(\mathbb{N})$  is denoted by  $\|\cdot\|_{n,\xi}$ . The set  $\widehat{\Xi}$  is Borel since the axioms defining a  $\mathbb{Q}(i)$ - $*$ -vector space are Borel conditions. The operator system  $\mathcal{OSy}(\xi)$  coded by  $\xi$  is the completion of  $S_\xi$ . Note in particular that the scalar multiplication can be uniquely extended on  $\mathcal{OSy}(\xi)$  from  $\mathbb{Q}(i)$  to  $\mathbb{C}$  and hence  $\mathcal{OSy}(\xi)$  is indeed an operator system.

**2.2.4. Parametrizations as models of a theory.** The Choi-Effros abstract characterization of operator systems allows one to describe operator systems as models of a theory  $\mathcal{T}_{\mathcal{OSy}}$  in a suitable language  $\mathcal{L}_{\mathcal{OSy}}$ . The details can be found in [27, Appendix B]. This allows one to define other parametrizations of operator systems as models of  $\mathcal{T}_{\mathcal{OSy}}$  as in [19, Subsection 3.4] or [7, Section 1], and as Polish structures as in [19, Subsections 2.1 and 3.3]. Lemma 2.3 of [19] together with the argument in [19, Subsection 3.4] show that these parametrizations are all weakly equivalent to each other. Moreover they can be easily seen to be equivalent to the parametrization  $\widehat{\Xi}$  defined above using again [19, Lemma 2.3]. The analogous argument for  $C^*$ -algebras is presented in [19, Section 3.1].

**2.3. Parametrizations of operator spaces.** One can define parametrizations for the category **OSp** of operator spaces in an analogous way for operator systems. To this purpose one can consider  $\mathcal{W}$  to be the  $\mathbb{Q}(i)$ -vector space of noncommutative polynomials in the variables  $X_i$  for  $i \in \mathbb{N}$  and with



no constant term. Fix an enumeration  $(q_n)_{n \in \mathbb{N}}$  of  $\mathcal{W}$ . The space  $\Gamma$  is just the set of sequences  $B(H)^{\mathbb{N}}$ . The space  $\Gamma_{\mathcal{W}}$  is the set of linear functions from  $\mathcal{W}$  to  $B(H)$ . It can be easily verified that  $\Gamma$  and  $\Gamma_{\mathcal{W}}$  are equivalent parametrizations.

The space  $\Xi$  is the Borel set of  $\delta = (\delta_n)_{n \in \mathbb{N}}$ , where  $\delta_n \in \mathbb{R}^{M_n(W)}$  is such that there is an operator system  $X$  and a nonzero sequence  $\gamma$  in  $X$  such that

$$\delta_n([p_{ij}]) = \|[p_{ij}(\gamma)]\|_{M_n(X)}$$

for every  $n \in \mathbb{N}$  and  $[p_{ij}] \in M_n(W)$ . One can describe  $\Xi$  as the set of  $\delta$  such that setting

$$\|[q_{k_{ij}}]\| = \delta_n([k_{ij}])$$

defines a nonzero *operator seminorm structure* on  $\mathcal{W}$ ; see [8]. This makes it apparent that  $\Xi$  is a Borel set. The operator space  $\mathcal{O}Sp(\delta)$  associated with  $\delta$  is the Hausdorff completion of  $\mathcal{W}$ —as described in [8]—with respect to the operator seminorm structure induced by  $\delta$ .

Let us say that a  $\mathbb{Q}(i)$ -vector space  $S$  is  $L^\infty$ -matricially normed if, for every  $n \in \mathbb{N}$ , the space of matrices  $M_n(S)$  is endowed with a norm  $\|\cdot\|_n$  such that the following hold.

- (1) For every  $n, m \in \mathbb{N}$ ,  $x \in M_n(S)$ , and  $y \in M_m(S)$

$$\left\| \begin{bmatrix} x & 0 \\ 0 & y \end{bmatrix} \right\|_{n,m} = \max \{ \|x\|_n, \|y\|_m \};$$

- (2) For every  $n, m, k \in \mathbb{N}$ ,  $a \in M_{n,k}(\mathbb{Q}(i))$ ,  $x \in M_k(S)$ , and  $b \in M_{k,m}(\mathbb{Q}(i))$ ,

$$\|axb\|_m \leq \|a\| \|x\|_k \|b\|$$

where  $\|a\|$  and  $\|b\|$  are the operator norms of  $a$  and  $b$  regarded as linear operators from  $\mathbb{C}^k$  to  $\mathbb{C}^n$  and from  $\mathbb{C}^m$  to  $\mathbb{C}^k$ .

Define  $\widehat{\Xi}$  to be the space of tuples

$$\xi = (f, g, (N_n)_{n \in \mathbb{N}}) \in \mathbb{N}^{\mathbb{N}^2} \times \mathbb{N}^{\mathbb{Q}(i) \times \mathbb{N}} \times \prod_{n \in \mathbb{N}} \mathbb{R}^{M_n(\mathbb{N})}$$

such that  $\xi$  codes an  $L^\infty$ -matricially normed  $\mathbb{Q}(i)$ -vector space structure  $S_\xi$  on  $\mathbb{N}$  by setting

$$\begin{aligned} n +_\xi m &= f_\xi(n, m), \\ q \cdot_g m &= g_\xi(q, m), \text{ and} \\ \|[m_{ij}]\| &= N_{n,m}((m_{ij})) \end{aligned}$$

where  $[m_{ij}]$  is an  $n \times m$  matrix. The fact that the axioms of a  $L^\infty$ -matricially normed  $\mathbb{Q}(i)$ -vector spaces are Borel conditions shows that  $\widehat{\Xi}$  is a Borel set. The operator space  $\mathcal{O}Sp(\xi)$  coded by  $\xi$  is the completion of  $S_\xi$ . The scalar multiplication on  $\mathcal{O}Sy(\xi)$  can be uniquely extended from  $\mathbb{Q}(i)$  to  $\mathbb{C}$  and hence  $\mathcal{O}Sp(\xi)$  is indeed an operator space.

Finally one can regard operator spaces as models of a theory in the logic for metric structures; see [27, Appendix B]. This gives other natural

parametrizations for the category of operator spaces as in [19, Subsection 3.3]. The same observations as the ones presented for operator systems in 2.2.4 apply.

### 3. THE CLASSIFICATION OF ALL SEPARABLE OPERATOR SYSTEMS AND SPACES

In this section we identify the complexity of the classification problem for operator spaces. The corresponding problem for separable operator systems has already been identified. By [19, Theorem 1.1] the complete order isomorphism relation of operator systems is Borel reducible to a Polish group action. And it follows from [40, Theorem 1.1] that such a relation is in fact maximal among all relations that lie below a Polish group action. (Observe that two  $C^*$ -algebras are  $*$ -isomorphic if and only if they are complete order isomorphic by [8, Corollary 1.3.10].) We now show that the classification of operator spaces up to completely bounded isomorphism is maximally complex among all analytic equivalence relations. This has been independently observed by N. Christopher Phillips (unpublished).

It is easy to see that the completely bounded isomorphism equivalence relation is analytic (say in the parameterization  $\Xi$ ). To show that it is maximal among analytic equivalence relations, we need only find a Borel reduction from another relation that is known to be complete. For this we will use the isomorphism relation on Banach spaces. This latter relation is defined on the standard Borel space  $\mathfrak{B}$  of closed subspaces of  $C[0, 1]$ , endowed with the Effros Borel structure. (Recall that any Banach space is isometrically isomorphic to a closed subspace of  $C[0, 1]$ .) It is shown in Theorem 5 of [25] that the isomorphism relation on  $\mathfrak{B}$  is indeed complete for analytic equivalence relations.

**Theorem 3.1.** *The classification problem for separable Banach spaces up to isomorphism is Borel reducible to the classification problem for separable operator spaces up to completely bounded isomorphism. As a consequence, the latter equivalence relation is complete analytic.*

*Proof.* Suppose that  $X \in \mathfrak{B}$  is a closed subspace of  $C[0, 1]$ . The minimal operator space structure on  $X$  is defined by

$$\|[x_{ij}]\| = \sup \{ \|\varphi(x_{ij})\| : \varphi \in X', \|\varphi\| \leq 1 \}.$$

Observe that by the Hahn–Banach theorem, it is equivalent to let  $\varphi$  range over a weak\*-dense subset of the unit ball of  $C[0, 1]'$ . It is well known that two Banach spaces  $X$  and  $Y$  are isomorphic if and only if they are completely isomorphic as operator spaces when endowed with their minimal operator space structures; see [8, 1.2.21]. Therefore the assignment  $X \mapsto X_{\min}$ , where  $X_{\min}$  is the minimal operator structure on  $X$ , is a reduction from the relation of isomorphism of Banach spaces to the relation of complete isomorphism of operator spaces. We need only verify that this reduction is given by a Borel function.

For this, fix a weak\*-dense subset  $D$  of the unit ball of  $C[0, 1]'$ . By the Kuratowski–Ryll–Nardzewski theorem [34, Theorem 12.13] there is a sequence of Borel functions  $\sigma_n : \mathfrak{B} \rightarrow C[0, 1]$  for  $n \in \mathbb{N}$  such that

$$\{\sigma_n(X) : n \in \mathbb{N}\}$$

is a dense subset of  $X$  for every  $X \in \mathfrak{B}$ . For each  $X \in \mathfrak{B}$  and  $q \in \mathcal{W}$  define  $q^X$  to be the element of  $X$  obtained from  $q$  replacing  $X_i$  with  $\sigma_i(X)$  for every  $i \in \mathbb{N}$ . We define the code  $\delta_X \in X$  for the operator space  $X_{\min}$  by setting

$$\delta_X([q_{ij}]) = \sup \{\|\varphi(q_{ij}^X)\| : \varphi \in D\}.$$

Then the function  $X \mapsto \delta_X$  from  $\mathfrak{B}$  to  $\Xi$  is Borel, as desired.  $\square$

#### 4. THE CLASSIFICATION OF FINITELY GENERATED OPERATOR SYSTEMS

We will show that, in contrast with the result from Section 3, the classification problem for finitely generated operator systems is smooth. For convenience we will work in the parametrization  $\Gamma$ . Fix  $N \in \mathbb{N}$  and denote by  $\Gamma_{\leq N}$  the set of  $\gamma \in \Gamma$  such that  $\mathcal{OSy}(\gamma)$  has dimension at most  $N$ . It will be shown in Appendix A that  $\Gamma_{\leq N}$  is a Borel subset of  $\Gamma$ . Denote by  $\Gamma_N$  the Borel set  $\Gamma_{\leq N} \setminus \Gamma_{\leq (N-1)}$ , which provides a standard Borel parametrization of operator systems of dimension  $N$ . Further define  $\widehat{\Gamma}_N$  to be the set of linearly independent tuples  $(x_1, \dots, x_N)$  such that  $\text{span}\{x_1, \dots, x_N\}$  is an operator system. It will be shown in Appendix A that  $\widehat{\Gamma}_N$  is a Borel subset of  $B(H)^N$ , and moreover the parametrizations  $\Gamma_N$  and  $\widehat{\Gamma}_N$  of  $N$ -dimensional operator systems are weakly equivalent.

**4.1. The classification of all finitely generated operator systems.** In this subsection we consider the classification problem for arbitrary finitely generated operator systems.

**Theorem 4.1.** *The relation of complete order isomorphism of finitely generated operator systems is smooth.*

Theorem 4.1 can be seen as a generalization of the main results of [4], where Arveson provided a concrete classification for operator systems with finite-dimensional  $C^*$ -envelope.

In order to prove Theorem 4.1 it is enough to show that for every  $N \in \mathbb{N}$  such relation is smooth when restricted to operator systems of dimension  $N$ . For convenience we will work in the parametrization for  $N$ -dimensional operator systems  $\widehat{\Gamma}_N$ . We proceed to define a compact metrizable space of complete isometry classes of  $N$ -dimensional operator systems.

The argument is analogous to the one for operator spaces from [38, Chapter 21]—see in particular Remark 21.2, Lemma 21.7, and Exercise 21.1 from [38]. Suppose that  $X$  and  $Y$  are  $N$ -dimensional operator systems with Archimedean matrix order units  $e_X$  and  $e_Y$ . Fix  $n \in \mathbb{N}$  and define  $d_n(X, Y)$  to be the infimum of

$$\max \{\|u(e_X) - e_Y\|, \log \|id_{M_n(\mathbb{C})} \otimes u\|, \log \|id_{M_n(\mathbb{C})} \otimes u^{-1}\|\}$$

where  $u$  ranges over all isomorphisms from  $X$  to  $Y$ .

**Lemma 4.2.** *If  $X$  and  $Y$  are  $N$ -dimensional operator systems, then  $X$  and  $Y$  are completely order isomorphic if and only if  $d_n(X, Y) = 0$  for every  $n \in \mathbb{N}$ .*

*Proof.* Necessity is obvious. Conversely suppose that  $d_n(X, Y) = 0$  for every  $n \in \mathbb{N}$ . Therefore for every  $n \in \mathbb{N}$  there is an isomorphism  $u_n : X \rightarrow Y$  such that

$$\|id_{M_n(\mathbb{C})} \otimes u_n\| < 1 + 2^{-n},$$

$$\|id_{M_n(\mathbb{C})} \otimes u_n^{-1}\| < 1 + 2^{-n},$$

and

$$\|u_n(e_X) - e_Y\| < 2^{-n}.$$

Fix a basis  $(x_1, \dots, x_N)$  for  $X$ . After passing to a subsequence we can assume that for every  $i \leq N$  the sequence  $(u_n(x_i))_{n \in \mathbb{N}}$  converges in  $Y$  to some  $y_i$ . Define  $u$  to be the linear function from  $X$  to  $Y$  sending  $x_i$  to  $y_i$  for every  $i \leq N$ . From the fact that a unital map is completely positive if and only if it is completely contractive, we deduce that  $u$  is a complete order isomorphism from  $X$  to  $Y$ .  $\square$

By Lemma 4.2 we can consider the space  $OSy(N)$  of complete order isomorphism classes of  $N$ -dimensional operator systems endowed with the topology induced by the metric

$$\delta_w(X, Y) = \sum_{n \in \mathbb{N}} 2^{-n} d_n(X, Y).$$

**Lemma 4.3.** *The space  $OSy(N)$  is compact.*

*Proof.* Suppose that  $(X_k)_{k \in \mathbb{N}}$  is a sequence of  $N$ -dimensional operator systems. By [14, Theorem 4.13] for every  $k \in \mathbb{N}$  one can find a normalized linear basis  $\bar{b}^{(k)}$  for  $X_k$  such that the dual basis is also normalized. Observe that this implies that

$$\|(\lambda_1, \dots, \lambda_N)\|_{\ell_N^1} = \sum_{i \leq N} |\lambda_i| \leq N \|(\lambda_1, \dots, \lambda_N)\|_{\ell_N^\infty} \leq N \left\| \sum_{i \leq N} \lambda_i b_i^{(k)} \right\|$$

for  $\lambda_1, \dots, \lambda_N \in \mathbb{C}$ . The basis  $\bar{b}^{(k)}$  induces a linear isomorphism of  $X_k$  with  $\mathbb{C}^N$ . Thus we can assume without loss of generality that the support of  $X_k$  is  $\mathbb{C}^N$ . Observe that  $\|x\|_k \leq \|x\|_{\ell_N^1}$  for every  $x \in X_k$ . Denote by  $\Omega$  the unit ball of  $\ell_N^1$  and let  $\bar{e}^{(k)}$  be the order unit of  $X_k$  for every  $k \in \mathbb{N}$ . The functions  $\|\cdot\|_k$  are equiuniformly continuous on  $\Omega$ . Therefore by the Arzelà-Ascoli theorem one can assume, after passing to a subsequence, that the sequence

$$(\|\cdot\|_k)_{k \in \mathbb{N}}$$

converges uniformly on  $\Omega$ , and moreover  $e_i^{(k)}$  converges for every  $i \leq n$ . Denote by  $\|\cdot\|_\infty$  and  $e_i^{(\infty)}$  the corresponding limits. If  $Y$  is the corresponding Banach space, then the construction ensures that  $X_k \rightarrow Y$  in the Banach Mazur distance for Banach spaces. A similar argument can be applied to the Banach spaces  $M_n(X_k)$  for every  $n \in \mathbb{N}$ , every time passing to a further subsequence. Finally take a diagonal subsequence and denote by  $X_\infty$  the  $L_\infty$ -matrix-norm structure on  $\mathbb{C}^N$  with distinguished element  $\bar{e}^{(\infty)}$  obtained as a limit. For every  $k \in \mathbb{N}$  fix a complete order embedding  $u_k$  of  $X_k$  into a unital  $C^*$ -algebra  $A_k$ . Fix a nonprincipal ultrafilter  $\mathcal{U}$  over  $\mathbb{N}$  and define  $A$  to be the ultraproduct  $\prod_{\mathcal{U}} A_k$ . Define the function  $u : \mathbb{C}^N \rightarrow A$  by

$$u(x) = \lim_{k \rightarrow \mathcal{U}} u_k(x).$$

It is not difficult to verify that  $u$  is a unital completely isometric embedding of  $X_\infty$  into  $A$ . This shows that  $X_\infty$  is an operator system, which is by construction the limit of the sequence  $(X_k)_{k \in \mathbb{N}}$ .  $\square$

It only remains to show that the function from  $\widehat{\Gamma}_N$  to  $OSy(N)$  assigning to  $\bar{x}$  the complete order isomorphism class of  $\text{span}(\bar{x})$  is Borel. Denote by  $\mathcal{W}_N$  the set of polynomials of degree 1 in the noncommutative variables  $X_1, \dots, X_N$  and with coefficients from  $\mathbb{Q}(i)$ . Similarly denote by  $\mathcal{W}_{\mathbb{C},N}$  the set of polynomials of degree 1 in the noncommutative variables  $X_1, \dots, X_N$  and with coefficients from  $\mathbb{C}$ .

**Lemma 4.4.** *There is a Borel function  $\bar{x} \mapsto p_{\bar{x}}$  from  $\widehat{\Gamma}_N$  to  $\mathcal{W}_{\mathbb{C},N}$  such that  $p_{\bar{x}}(\bar{x}) = I$ .*

*Proof.* Fix  $k \in \mathbb{N}$ . Denote by  $\widehat{\Gamma}_{N,k}$  the Borel set of  $\bar{x} \in \widehat{\Gamma}_N$  such that for every  $\varepsilon > 0$  there is  $p \in \mathcal{W}_N$  such that  $\|p\| \leq k$  and  $\|p(x_1, \dots, x_N) - I\| < \varepsilon$ . Observe that the relation

$$\{(\bar{x}, p) \in \widehat{\Gamma}_{N,k} \times \mathcal{W}_{\mathbb{C},N} : p(\bar{x}) = I, \|p\| \leq k\}$$

is Borel and has compact sections. Then the conclusion follows from the Kuratowski–Ryll–Nardzewski theorem [34, Theorem 12.13].  $\square$

We can finally present the proof of Theorem 4.1.

*Proof of Theorem 4.1.* As observed earlier, it is enough to show that for every  $N \in \mathbb{N}$  the relation of complete order isomorphism of  $N$ -dimensional operator systems is smooth. For convenience we will work in the parametrization  $\widehat{\Gamma}_N$ . Consider the compact metrizable space  $OSy(N)$  defined above. The map from  $\widehat{\Gamma}_N$  to  $OSy(N)$  that assigns to an element  $(x_1, \dots, x_N)$  of  $\widehat{\Gamma}_N$  the class of  $\text{span}(\bar{x})$  is a reduction from the relation of complete order isomorphism to the relation of equality in  $OSy(N)$ . In order to conclude that the former relation is smooth it is enough to show that such a reduction is Borel. To this purpose, fix  $\bar{x} \in \widehat{\Gamma}_N$ ,  $n \in \mathbb{N}$ , and  $\varepsilon > 0$ . It is enough to

show that the set of  $\bar{y} \in \widehat{\Gamma}_N$  such that

$$d_n(\text{span}(\bar{x}), \text{span}(\bar{y})) < \varepsilon$$

is Borel. Observe  $\bar{y}$  belongs to such a set if and only if there are  $p_i \in \mathcal{W}_N$  such that

- (1)  $\|p_e^{\bar{x}}(\bar{y}) - I\| < \varepsilon$ ;
- (2)  $\|[p_{ij}(\bar{y})]\| \leq (1 + \varepsilon) \|[p_{ij}(x)]\|$  for every  $p_{ij} \in \mathcal{W}_N$ ;
- (3)  $\|[p_{ij}(x)]\| \leq (1 + \varepsilon) \|[p_{ij}(\bar{y})]\|$  for every  $p_{ij} \in \mathcal{W}_N$ .

By virtue of Lemma 4.4 these are Borel conditions.  $\square$

**4.2. Operator systems generated by a single unitary.** In this subsection we provide a concrete classification of operator systems generated by a single unitary operator  $U$  in terms of the spectrum  $\sigma(U)$  of  $U$ . First of all we observe that the set  $\Gamma_U$  of  $\gamma \in \Gamma$  such that  $\mathcal{OSy}(\gamma)$  is generated by a single unitary is Borel. In fact  $\gamma \in \Gamma_U$  if and only if  $\gamma \in \Gamma_{\leq 3}$  and moreover there is  $n \in \mathbb{N}$  such that for every  $m \in \mathbb{N}$  and  $q_1, \dots, q_m \in \mathcal{V}$  there is  $p \in \mathcal{V}_n$  such that  $\|p\| \leq n$ ,  $\|(p^*p)(\gamma) - 1\| < \frac{1}{m}$  and for every  $j \leq m$  there is  $r \in \mathcal{V}$  such that

$$\|r(p(\gamma)) - q_j(\gamma)\| < 1/m.$$

This argument together with [34, Theorem 28.8] shows that there is a Borel map  $\gamma \mapsto p^\gamma$  from  $\Gamma_U$  to  $\mathcal{V}_{\mathbb{C}}$  such that  $p^\gamma(\gamma)$  is a unitary generator of  $\Gamma_U$ . This also shows that the standard Borel space  $\mathcal{U}(H)$  of unitary operators on  $H$  is a weakly equivalent parametrization for the category of operator systems generated by a single unitary. (Observe that  $\mathcal{U}(H)$  is a  $G_\delta$  subset of the unit ball of  $B(H)$  with respect to the weak operator topology, and hence a standard Borel space.)

In the following we will consider the parametrization  $\mathcal{U}(H)$ . The operator system  $\mathcal{OSy}(U)$  coded by  $U$  is the closed linear span of the set  $\{1, U, U^*\}$ . Recall that the  $C^*$ -envelope of an operator system is in some sense the minimal  $C^*$ -algebra containing a given operator system. It was introduced in terms of boundary representations in [1]. The *Šilov ideal* of an operator system is the intersection of the kernels of all of its boundary representations; when this ideal is trivial, we say that the system is *reduced*, and the  $C^*$ -algebra it generates is its  $C^*$ -envelope.

An abstract proof of the existence of the  $C^*$ -envelope, making no reference to boundary representations, was given by Hamana using his theory of injective envelopes [28, 29]. The existence of sufficiently many boundary representations to determine the  $C^*$ -envelope was finally established in the separable case in [3] and in full generality in [16]. Combining this with Arveson's initial work we have the following:

**Theorem 4.5.** *Let  $\mathcal{S}_1 \subset C^*(\mathcal{S}_1)$  and  $\mathcal{S}_2 \subset C^*(\mathcal{S}_2)$  be two reduced operator systems, and let  $\phi : \mathcal{S}_1 \rightarrow \mathcal{S}_2$  be a complete order isomorphism. Then there exists a  $*$ -isomorphism  $\tilde{\phi} : C^*(\mathcal{S}_1) \rightarrow C^*(\mathcal{S}_2)$ , with  $\tilde{\phi}|_{\mathcal{S}_1} = \phi$ .*

*Proof.* This follows from [1, Theorem 2.2.5] and [16, Theorem 3.4].  $\square$

For the reader's convenience, we include the following well-known fact.

**Lemma 4.6.** *Suppose that  $V_1, \dots, V_n \in B(H)$ . Define  $X$  to be the operator system generated by*

$$\{V_1, \dots, V_n, V_1^*V_1, \dots, V_n^*V_n, V_1V_1^*, \dots, V_nV_n^*\}.$$

*Then the  $C^*$ -envelope of  $X$  can be identified with the  $C^*$ -algebra  $C^*(X)$  generated by  $X$  inside  $B(H)$ .*

*Proof.* We will show that every (irreducible) representation of  $C^*(X)$  is a boundary representation, so the Šilov ideal of  $X$  is trivial, implying the result. We actually show that  $X$  is hyperrigid, but we don't really need this fact. Let  $\pi : C^*(X) \rightarrow B(H)$  be a representation. We must show that if  $\phi : C^*(X) \rightarrow B(H)$  is a completely positive extension of the restriction  $\pi|_X$ , then  $\phi = \pi$ . We have for every  $i = 1, \dots, n$ ,

$$\phi(V_i^*)\phi(V_i) = \pi(V_i^*)\pi(V_i) = \pi(V_i^*V_i) = \phi(V_i^*V_i),$$

and similarly  $\phi(V_i)\phi(V_i^*) = \phi(V_iV_i^*)$ . Thus  $V_1, \dots, V_n$  each belong to the multiplicative domain of  $\phi$  (see e.g. [8, Proposition 1.3.11]), and so  $\phi$  is multiplicative. Since  $\phi(V_i) = \pi(V_i)$  for each  $i = 1, \dots, n$ , it follows that  $\phi = \pi$ .  $\square$

We can now address the announced classification of operator systems generated by a single unitary.

**Theorem 4.7.** *Suppose that  $U, V \in \mathcal{U}(H)$  and  $\sigma(V)$  has at least 5 points. Then the following statements are equivalent:*

- (1) *there exists a  $*$ -isomorphism  $\pi : C^*(U) \rightarrow C^*(V)$  with  $\pi(U) = \lambda V$  or  $\pi(U) = \lambda V^*$  for some  $\lambda \in \mathbb{T}$ ;*
- (2)  *$\sigma(U) = \lambda\sigma(V)$  or  $\sigma(U) = \lambda\overline{\sigma(V)}$  for some  $\lambda \in \mathbb{T}$ ;*
- (3)  *$\mathcal{OSy}(U)$  is completely order isomorphic to  $\mathcal{OSy}(V)$ .*

*If  $\sigma(V)$  has 3 points or less, then  $\mathcal{OSy}(U)$  is completely order isomorphic to  $\mathcal{OSy}(V)$  if and only if  $|\sigma(U)| = |\sigma(V)|$ .*

*Proof.* The implication (1)  $\implies$  (2) is clear by the spectral mapping theorem. For (2)  $\implies$  (3), let  $W = \lambda V$ . Then the hypothesis is that  $\sigma(U) = \sigma(W)$ . Thus  $C^*(U) \simeq C(\sigma(U)) = C(\sigma(W)) \simeq C^*(W)$  as  $C^*$ -algebras, via a  $*$ -isomorphism  $\pi$  such that  $\pi(U) = W$  (because the two isomorphisms above map  $U \mapsto z \mapsto W$ ). If we let  $\varphi = \pi|_{\mathcal{OSy}(U)}$ , we get a unital completely positive map  $\varphi : \mathcal{OSy}(U) \rightarrow \mathcal{OSy}(W)$  with

$$\varphi(\alpha I + \beta U + \gamma U^*) = \alpha I + \beta W + \gamma W^*.$$

This is well-defined and bijective, because of the fact that  $|\sigma(U)| = |\sigma(W)| = |\sigma(V)| > 2$  implies that  $I, U, U^*$  are linearly independent. We have  $\varphi^{-1} = \pi^{-1}|_{\mathcal{OSy}(V)}$ , so it is also unital completely positive. Thus,  $\mathcal{OSy}(U)$  is completely order isomorphic to  $\mathcal{OSy}(W) = \mathcal{OSy}(V)$ . Finally, for the case where  $\sigma(U) = \lambda\overline{\sigma(V)}$ , we take  $W = \lambda V^*$  and repeat the argument above.

Now we show that (3)  $\implies$  (1). Suppose that  $\mathcal{OSy}(U)$  is completely order isomorphic to  $\mathcal{OSy}(V)$ . By Lemma 4.6 the  $C^*$ -envelope  $C_e^*(\mathcal{OSy}(U))$  coincides with the  $C^*$ -algebra  $C^*(U)$  generated by  $U$  inside  $B(H)$ . The same applies to  $V$ . By Theorem 4.5, there is a  $*$ -isomorphism  $\varphi$  from  $C^*(U)$  to  $C^*(V)$  that sends  $\mathcal{OSy}(U)$  onto  $\mathcal{OSy}(V)$ . In particular

$$\varphi(U) = \alpha I + \beta V + \gamma V^*$$

for some  $\alpha, \beta, \gamma \in \mathbb{C}$ . Since  $U$  is unitary and  $\varphi$  is a  $*$ -homomorphism we have that

$$\begin{aligned} I = \varphi(U^*U) &= (|\alpha|^2 + |\beta|^2 + |\gamma|^2) I + (\bar{\alpha}\beta + \alpha\bar{\gamma}) V + (\alpha\bar{\beta} + \bar{\alpha}\gamma) V^* \\ &\quad + \bar{\beta}\gamma V^2 + \beta\bar{\gamma}(V^*)^2. \end{aligned}$$

The assumption that  $\sigma(V)$  has at least 5 points implies that the elements  $\{I, V, V^*, V^2, (V^*)^2\}$  of  $B(H)$  are linearly independent (because  $C^*(V)$  is at least 5-dimensional). Therefore we get the relations

$$\begin{aligned} 1 &= |\alpha|^2 + |\beta|^2 + |\gamma|^2, \\ 0 &= \bar{\alpha}\beta + \alpha\bar{\gamma}, \\ 0 &= \alpha\bar{\beta} + \bar{\alpha}\gamma, \\ 0 &= \bar{\beta}\gamma. \end{aligned}$$

Exactly one between  $\beta$  and  $\gamma$  is zero (if both were zero, we would get  $\varphi(U) = \alpha I$ , and the image of  $\varphi$  would be one-dimensional). This forces  $\alpha = 0$  and  $|\beta| = 1, \gamma = 0$ ; or  $\beta = 0$  and  $|\gamma| = 1$ .

Finally, if  $\sigma(U)$  has at most 3 points then  $\mathcal{OSy}(U) = C_e^*(\mathcal{OSy}(U))$ . So the complete order isomorphism of the operator systems agrees with the isomorphism of the  $C^*$ -algebras, which is given by homeomorphism of the spectra.  $\square$

*Remark 4.8.* Denote by  $\mathcal{K}(\mathbb{T})$  the space of closed subsets of  $\mathbb{T}$  endowed with the Vietoris topology [34, 4.F]. By [35, Theorem 1.1] the function  $U \mapsto \sigma(U)$  assigning to  $U \in \mathcal{U}(H)$  its spectrum is Borel. Denote by  $\text{Isom}(\mathbb{T})$  the group of isometries of  $\mathbb{T}$  endowed with the topology of pointwise convergence. Elements of  $\text{Isom}(\mathbb{T})$  are of the form  $z \mapsto \lambda z$  or  $z \mapsto \lambda \bar{z}$  for  $\lambda \in \mathbb{T}$ . The group of isometries of  $\mathbb{T}$  has a natural continuous action on  $\mathcal{K}(\mathbb{T})$ . Denote by  $E_{\text{Iso}(\mathbb{T})}^{\mathcal{K}(\mathbb{T})}$  the corresponding orbit equivalence relation. Observe that, being the orbit equivalence relation of a continuous action of a compact group,  $E_{\text{Iso}(\mathbb{T})}^{\mathcal{K}(\mathbb{T})}$  is in particular smooth. Theorem 4.7 shows that  $U \mapsto \sigma(U)$  is a Borel reduction from the relation of complete order isomorphism of operator systems generated by a single unitary with at least 5 points in the spectrum to  $E_{\text{Iso}(\mathbb{T})}^{\mathcal{K}(\mathbb{T})}$ .

We do not know exactly what happens when  $\sigma(U)$  has exactly 4 points. The following example at least shows that the operator systems  $\mathcal{OSy}(U)$  for



$U$  unitary with four-point spectrum are not all isomorphic, so the behaviour seems to be more similar to the 5+ case. Consider the unitary elements

$$U = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & i & 0 \\ 0 & 0 & 0 & -i \end{bmatrix}, \quad V = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1+i}{\sqrt{2}} & 0 & 0 \\ 0 & 0 & i & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

of  $M_4(\mathbb{C})$ . Then  $C_e^*(\mathcal{OSy}(U)) = C^*(U) = C^*(V) = C_e^*(\mathcal{OSy}(V))$  is the diagonal MASA of  $M_4(\mathbb{C})$ . Suppose by contradiction that  $\mathcal{OSy}(U) \cong \mathcal{OSy}(V)$ . Then Theorem 4.5 there is a  $*$ -isomorphism  $\pi$  from  $C^*(U)$  onto  $C^*(V)$  mapping  $\mathcal{OSy}(U)$  onto  $\mathcal{OSy}(V)$ . Then  $\pi(U) \in \mathcal{OSy}(V)$  is a diagonal matrix with eigenvalues  $\{1, -1, i, -i\}$  that is in the span of  $\{I, V, V^*\}$ . Consider the determinant

$$\begin{vmatrix} a & 1 & 1 & 1 \\ b & 1 & \frac{1+i}{\sqrt{2}} & \frac{1-i}{\sqrt{2}} \\ c & 1 & i & -i \\ d & 1 & -1 & -1 \end{vmatrix} = 2i(-a + 2b - \sqrt{2}c + (\sqrt{2} - 1)d).$$

Observe that no choice of  $a, b, c, d$  with  $\{a, b, c, d\} = \{1, -1, i, -i\}$  can make the above determinant equal to zero. This shows that  $\pi(U)$  is linearly independent from  $I, V, V^*$ , a contradiction.

**4.3. Uncountably many classes.** We now show that the relation of complete order isomorphism of finitely generated operator systems has uncountably many classes. In fact there are uncountably many classes even when restricting to operator systems whose  $C^*$ -envelope is  $M_3(\mathbb{C})$ . For  $t \in (0, 1]$  consider the operator

$$W_t = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & t & 0 \end{bmatrix}.$$

Observe that  $W_t$  is irreducible and hence the  $C^*$ -algebra  $C^*(W_t)$  generated by  $W_t$  coincides with  $M_3(\mathbb{C})$ . The  $C^*$ -envelope can always be realized as a quotient of  $C^*(W_t)$ . By simplicity of  $M_3(\mathbb{C})$  we must have  $C_e^*(W_t) = C^*(W_t) = M_3(\mathbb{C})$ . Denote by  $X_t$  the operator system  $\mathcal{OSy}(W_t)$  generated by  $W_t$ . We claim that  $X_t$  and  $X_s$  are complete order isomorphic if and only if  $s = t$ . Suppose that  $X_t$  and  $X_s$  are complete order isomorphic. As mentioned above, the simplicity of  $M_3(\mathbb{C})$  makes both  $X_t$  and  $X_s$  to be reduced. By Theorem 4.5, there is a  $*$ -automorphism of  $M_3(\mathbb{C})$  that maps  $X_t$  onto  $X_s$ . It is well-known that  $*$ -automorphisms of finite-dimensional  $C^*$ -algebras are implemented by unitaries, so there is a unitary  $U \in M_3(\mathbb{C})$  such that the function  $\Phi : A \mapsto UAU^*$  sends  $X_t$  onto  $X_s$ . In particular

$$\Phi(W_t) = \alpha I + \beta W_s + \gamma W_s^*$$

for some  $\alpha, \beta, \gamma \in \mathbb{C}$ . Denote by  $\tau$  the trace on  $M_3(\mathbb{C})$ . Since  $\Phi$  is trace-preserving we have that

$$0 = \tau(W_t) = \tau(\Phi(W_t)) = \alpha.$$

Also

$$0 = \tau(W_t^2) = \tau(\Phi(W_t)^2) = 2(1 + s^2)\beta\gamma.$$

Therefore exactly one between  $\beta$  and  $\gamma$  is zero (since  $W_t \neq 0$ ). Suppose that  $\gamma = 0$ . In this case, the singular values of  $W_t$  are  $\{0, 1, t\}$  and the singular values of  $\Phi(W_t) = \beta W_s$  are  $\{0, |\beta|, |\beta|s\}$ . So either  $|\beta| = 1$  and  $s = t$ , or  $|\beta| = t$  and  $|\beta|s = 1$ . As both  $|\beta| \leq 1$  and  $s \leq 1$ , this requires  $|\beta| = s = 1$ , and then  $t = |\beta| = 1 = s$ . In a similar way we can conclude  $s = t$  in the case  $\beta = 0$ .

Therefore  $(X_t)_{t \in (0,1]}$  is a one-parameter family of pairwise not completely order isomorphic operator systems with the same  $C^*$ -envelope  $M_3(\mathbb{C})$ .

Using arguments similar to those above—just a little more involved—we can also produce an uncountable family of non-isomorphic operator systems with  $C^*$ -envelope  $M_2(\mathbb{C})$ , by taking  $W_t = \begin{bmatrix} 1 & 0 \\ t & 0 \end{bmatrix}$ .

**4.4. Compact subsets of  $\mathbb{C}^n$ .** Denote by  $\mathcal{K}(\mathbb{C})$  the space of compact subsets of  $\mathbb{C}$  endowed with the Effros Borel structure. It can be essentially deduced from results of [30] that the relation of homeomorphism of compact subsets of  $\mathbb{C}$  is not classifiable by countable structures. (This is an unpublished observation of Farah-Toms-Törnquist.) Moreover it is an open problem whether this relation has in fact maximal complexity among all the relations that are classifiable by the orbits of a continuous action of a Polish group on a Polish space. In this direction, it was recently shown by Zielinski that the relation of homeomorphism of compact metrizable spaces has indeed maximal complexity [42].

In this subsection we consider another related equivalence relation on  $\mathcal{K}(\mathbb{C})$ .

**Definition 4.9.** Suppose that  $D, \tilde{D}$  are closed subsets of  $\mathbb{C}$ . A *degree 1 map* (in  $z$  and  $\bar{z}$  separately) on  $D$  is a function  $f$  from  $D$  to  $\mathbb{C}$  of the form

$$(1) \quad f(z) = \alpha + \beta z + \gamma \bar{z} + \delta z \bar{z}$$

for some  $\alpha, \beta, \gamma, \delta \in \mathbb{C}$  such that moreover for some  $\alpha', \beta', \gamma', \delta' \in \mathbb{C}$ ,

$$(2) \quad f(z) \overline{f(z)} = \alpha' + \beta' z + \gamma' \bar{z} + \delta' z \bar{z}$$

for every  $z \in D$ . A *degree 1 homeomorphism* from  $D$  to  $\tilde{D}$  is a homeomorphism  $\varphi : D \rightarrow \tilde{D}$  such that  $\varphi$  and  $\varphi^{-1}$  are degree 1 maps. The closed subsets  $D$  and  $\tilde{D}$  of  $\mathbb{C}$  are *degree 1 homeomorphic* if there is a degree 1 homeomorphism from  $D$  to  $\tilde{D}$ .

The natural definition of degree 1 map in  $z$  and  $\bar{z}$  would only involve Condition 1. However, Condition 2 is necessary to ensure that the relation

of degree 1 homeomorphism be an equivalence relation. In the rest of this subsection we will prove Theorem 4.12 below.

We need to relate degree 1 homeomorphism with complete order isomorphism of operator systems. Suppose that  $V, \tilde{V}$  are bounded normal linear operators on  $X$ . Define  $X = \mathcal{OSy}(V, VV^*)$ ,  $\tilde{X} = \mathcal{OSy}(\tilde{V}, \tilde{V}\tilde{V}^*)$ . Denote by  $A$  and  $\tilde{A}$  the (commutative)  $C^*$ -algebras  $C^*(V)$  and  $C^*(\tilde{V})$ , and denote by  $D$  and  $\tilde{D}$  their spectra (which coincide with the spectra of  $V$  and  $\tilde{V}$ ). The proof of the following lemma is immediate.

**Lemma 4.10.** *Under the identifications  $A \cong C(D)$  and  $\tilde{A} \cong C(\tilde{D})$ , isomorphisms from  $A$  to  $\tilde{A}$  mapping  $X$  onto  $\tilde{X}$  correspond to degree 1 homeomorphisms from  $D$  to  $\tilde{D}$ .*

Using Lemma 4.10 we can relate degree 1 homeomorphism of compact sets with complete order isomorphism of operator systems.

**Lemma 4.11.** *With the notations above, the following statements are equivalent:*

- (1)  $X$  and  $\tilde{X}$  are completely order isomorphic;
- (2) there is an isomorphism from  $A$  to  $\tilde{A}$  mapping  $X$  onto  $\tilde{X}$ ;
- (3)  $D$  and  $\tilde{D}$  are degree 1 homeomorphic.

*Proof.* By Lemma 4.6 the  $C^*$ -envelopes of  $X$  and  $\tilde{X}$  can be identified with  $A$  and  $\tilde{A}$ . Therefore the equivalence of (1) and (2) follows from Theorem 4.5. The equivalence of (2) and (3) is a consequence of Lemma 4.10.  $\square$

Now we can state and prove the aforementioned result.

**Theorem 4.12.** *The relation of degree 1 homeomorphism of subsets of  $\mathbb{C}$  is smooth.*

*Proof.* Working in the parametrization  $\Xi$ , it is clear that there is a Borel function from  $\mathcal{K}(\mathbb{C})$  to  $\Xi$  that assigns to a compact subset  $X$  of  $\mathbb{C}$  a code  $\xi_X$  for the operator system generated by the identity function inside the  $C^*$ -algebra  $C(X)$  of continuous complex-valued functions on  $X$ . By Lemma 4.11 this is a Borel reduction from the relation of degree 1 homeomorphism of compact subsets of  $\mathbb{C}$  to the relation of complete order isomorphism of finitely generated operator systems. In view of Theorem 4.1 in particular this shows that the relation of degree 1 homeomorphism of compact subsets of  $\mathbb{C}$  is smooth.  $\square$

We can also consider a natural generalization to compact subsets of  $\mathbb{C}^n$ .

**Definition 4.13.** Suppose that  $D, \tilde{D}$  are closed subsets of  $\mathbb{C}^n$ . Set  $z_0 = 1$ . A degree 1 map on  $D$  is a function from  $D$  to  $\mathbb{C}^n$  of the form

$$(z_1, \dots, z_n) \mapsto (f_1(z_1, \dots, z_n), \dots, f_n(z_1, \dots, z_n))$$

where for  $1 \leq k \leq n$

$$f_k(z_1, \dots, z_n) = \sum_{i,j=0}^n \beta_{ij}^{(k)} z_i \bar{z}_j$$

for some  $\beta_{ij}^{(k)} \in \mathbb{C}$  such that moreover for every  $1 \leq k, m \leq n$ ,

$$f_k(z_1, \dots, z_n) \overline{f_m(z_1, \dots, z_n)} = \sum_{i,j=0}^n \beta_{ij}^{(k,m)} z_i \bar{z}_j$$

for some  $\beta_{ij}^{(k,m)} \in \mathbb{C}$ . A *degree 1 homeomorphism* from  $D$  to  $\tilde{D}$  is a homeomorphism  $\varphi : D \rightarrow \tilde{D}$  such that  $\varphi$  and  $\varphi^{-1}$  are degree 1 maps. The closed subsets  $D$  and  $\tilde{D}$  of  $\mathbb{C}^n$  are *degree 1 homeomorphic* if there is a degree 1 homeomorphism from  $D$  to  $\tilde{D}$ .

Again this is an equivalence relation for compact subsets of  $\mathbb{C}^n$ .

**Theorem 4.14.** *The relation of degree 1 homeomorphism of subsets of  $\mathbb{C}^n$  is smooth.*

*Proof.* We work along a similar path as the proof of Theorem 4.12, by considering tuples  $(V_1, \dots, V_n)$  of pairwise commuting bounded normal linear operators on  $H$  and the operator system  $X$  generated by  $V_1, \dots, V_n$  and  $V_i V_j^*$  for  $1 \leq i, j \leq n$ . It is a consequence of Lemma 4.6 that the  $C^*$ -envelope of  $X$  can be identified with  $C^*(V_1, \dots, V_n)$ . Therefore the proof of Lemma 4.11 shows that two such operator systems are completely order isomorphic if and only if their spectra are degree 1 homeomorphic. Working in the parametrization  $\Xi$  one can easily see that there is a Borel map assigning to a compact subset  $D$  of  $\mathbb{C}^n$  the operator system generated by  $V_1, \dots, V_n$  and  $V_i V_j^*$  for  $1 \leq i, j \leq n$ , where  $V_1, \dots, V_n \in C(D)$  are the coordinate projections. The proof is concluded by observing that  $C(D)$  is generated by  $V_1, \dots, V_n$  as a  $C^*$ -algebra by the Stone-Weierstrass theorem.  $\square$

## 5. SMOOTH CLASSIFICATION OF STRUCTURES

In this section we isolate the model-theoretic content of Theorem 4.1 and show that the isometric classification of any class of structures that are compact or at least locally compact will be smooth. We use the logic for metric structures as defined in [20, Section 2.1]. In particular we consider possibly unbounded metric structures, endowed with suitable *domains of quantifications*. The allowed quantifiers are  $\sup_{x \in D}$  and  $\inf_{x \in D}$  for some domain of quantification  $D$ . The more usual framework of logic for metric structures from [6] only considers structures with metric bounded by 1 where the only domain of quantification is the whole structure.

Let  $\mathcal{L}$  be a countable language in this logic, and enumerate its domains of quantification  $(D_n)_{n \in \mathbb{N}}$ . We can suppose that  $\mathcal{L}$  contains only relation symbols; otherwise we replace each function symbol with a relation symbol

to be interpreted as the graph of the function. We can also suppose that  $\mathcal{L}$  has just one sort; the multi-sorted version of our result may be easily obtained by a suitable adaptation of the argument.

We work in the parametrization for separable  $\mathcal{L}$ -structures similar to the one considered in [7]. Write  $\mathbb{N}$  as an increasing union  $\bigcup_n Q_n$  of infinite sets such that  $Q_{n+1} \setminus Q_n$  is infinite for every  $n \in \mathbb{N}$ . An  $\mathcal{L}$ -structure is regarded as an element of  $\prod_B \mathbb{R}^{|B|}$  where  $B$  varies over all the relation symbols in  $\mathcal{L}$  (including the symbol for the metric  $d$ ) and  $|B|$  is the arity of  $B$ . Any element  $f = (f_B)_B$  of  $\prod_B \mathbb{R}^{|B|}$  with the right uniform continuity moduli codes and  $\mathcal{L}$ -structure  $M$  that has as support the completion of  $\mathbb{N}$  with respect to the metric  $f_d$  on  $\mathbb{N}$ . The interpretation of the domain of quantification  $D_n$  is the closure inside  $M$  of  $Q_n$ . Finally the interpretation of a relation symbol  $B$  is obtained by considering the unique (uniformly) continuous extension of  $f_B$  to  $M^{|B|}$ . As observed in [7] the set of tuples  $f_B \in \mathbb{R}^B$  that code an  $\mathcal{L}$ -structure is a Borel subset of  $\prod_B \mathbb{R}^{|B|}$  and hence a standard Borel space.

**Definition 5.1.** An  $\mathcal{L}$ -structure  $M$  is *proper* if the interpretation  $D^M$  of every domain of quantification is compact.

It is easy to check that the set  $\text{Mod}_p(\mathcal{L})$  of codes for proper  $\mathcal{L}$ -structures is a Borel. In fact  $(f_B) \in \text{Mod}_p(\mathcal{L})$  if and only if  $(f_B)$  is a code for an  $\mathcal{L}$ -structure and moreover  $\forall n, k \in \mathbb{N} \exists m \in \mathbb{N} \exists x_1, \dots, x_m \in Q_n$  such that  $\forall x \in Q_n \exists i \leq n$  such that  $d_B(x, x_i) < 2^{-k}$ .

Following [5, Definition 1.1] we say that a function  $f$  from a metric space  $X$  to  $[0, +\infty]$  is Katětov if

$$|f(x) - f(y)| \leq d(x, y) \leq f(x) + f(y)$$

for every  $x, y \in X$ . A function  $\psi : X \times Y \rightarrow [0, +\infty]$  is an *approximate isometry* from  $X$  to  $Y$  if it is separately Katětov in each argument. Equivalently an approximate isometry from  $X$  to  $Y$  is a code a metric on the disjoint union of  $X$  and  $Y$  that extends the given metrics on  $X$  and  $Y$  obtained by setting  $\widehat{d}(x, y) = \psi(x, y)$  for  $x \in X$  and  $y \in Y$ . If  $\psi$  is an approximate isometry from  $X$  to  $Y$  we write  $\psi : X \rightsquigarrow Y$ . If  $X_0$  and  $Y_0$  are subspaces of  $X$  and  $Y$  and  $\psi : X \rightsquigarrow Y$ , then one can consider the restriction-truncation  $\psi : X_0 \rightsquigarrow Y_0$  which is just the restriction of  $\psi$  to  $X_0 \times Y_0$ . An approximate isometry is  $\varepsilon$ -bijection if for every  $r > \varepsilon$  and every  $x \in X$  there is  $y \in Y$  such that  $\psi(x, y) < r$  and for every  $y \in Y$  there is  $x \in X$  such that  $\psi(x, y) < r$ . In the following we will use other notations and conventions from [5, Definition 1.1] regarding approximate isometries. Suppose now that  $M, N$  are  $\mathcal{L}$ -structures,  $\psi : M \rightsquigarrow N$ , and  $B \in \mathcal{L}$  is an  $n$ -ary relation symbol. Identify the interpretation  $B^M$  with its graph, and regard it as a metric space endowed with the max-metric. Define the approximate isometry  $\psi^B : B^M \rightsquigarrow B^N$  by

$$\psi^B(\bar{x}, B(\bar{x}), \bar{y}, B(\bar{y})) = \max \{ \psi(x_1, y_1), \dots, \psi(x_n, y_n), |B(\bar{x}) - B(\bar{y})| \}.$$

**Definition 5.2.** Suppose that  $M, N$  are  $\mathcal{L}$ -structures. Fix  $\varepsilon > 0$ ,  $k \in \mathbb{N}$ , a domain of quantification  $D$  in  $\mathcal{L}$  and a subset  $\mathcal{L}_0$  of  $\mathcal{L}$ . An  $(\mathcal{L}_0, \varepsilon)$ -approximate isomorphism from  $M$  to  $N$  is an approximate isometry  $M \rightsquigarrow N$  such that  $\psi^B$  is an  $\varepsilon$ -isometry for every  $B \in \mathcal{L}_0$ . A  $(D, \mathcal{L}_0, \varepsilon)$ -approximate isomorphism from  $M$  to  $N$  is an approximate isometry  $\psi : M \rightsquigarrow N$  such that the restriction-truncation  $\psi : D_k(M) \rightarrow D_k(N)$  is an  $(\mathcal{L}_0, \varepsilon)$ -approximate isomorphism from  $D(M)$  to  $D(N)$ .

Write the language  $\mathcal{L}$  as a countable increasing union of finite languages  $\mathcal{L}_k$  for  $k \in \mathbb{N}$  such that the metric symbol  $d$  belongs to  $\mathcal{L}_1$ . If  $k \in \mathbb{N}$  and  $M, N$  are  $\mathcal{L}$ -structures define  $d_k(M, N)$  to be the infimum of  $\varepsilon > 0$  such that there is a  $(D_k, \mathcal{L}_k, \varepsilon)$ -approximate isomorphism from  $M$  to  $N$ . The boundedness requirement on the values of the interpretations of relation symbols in  $\mathcal{L}$ -structures shows that  $d_k(M, N)$  is finite. Define then the Gromov–Hausdorff distance

$$d_{GH}(M, N) = \sum_{k \in \mathbb{N}} 2^{-k} d_k(M, N).$$

Recall the definition of formula from [20, Section 2.4]. A formula is *universal* if it only uses the quantifier  $\sup$ . Similarly a formula is *existential* if it only uses the quantifier  $\inf$ . For every  $n \in \mathbb{N}$  let us fix a uniformly dense countable set  $\mathcal{F}_n$  of functions from the interval  $[-n, n]$  to itself. A formula is *restricted* if it only uses connectives from  $\bigcup_n \mathcal{F}_n$ . Observe that there are only countably many restricted  $\mathcal{L}$ -formulas. (Recall that we are assuming  $\mathcal{L}$  to be countable.)

**Proposition 5.3.** *Suppose that  $M$  and  $N$  are two proper  $\mathcal{L}$ -structures. The following statements are equivalent:*

- (1)  $\varphi^M = \varphi^N$  for every restricted universal sentence  $\varphi$ ;
- (2)  $\varphi^M = \varphi^N$  for every universal sentence  $\varphi$ ;
- (3)  $\varphi^M = \varphi^N$  for every existential sentence  $\varphi$ ;
- (4)  $d_{GH}(M, N) = 0$ ;
- (5)  $M$  and  $N$  are isomorphic;
- (6)  $M$  and  $N$  are bi-embeddable.

*Proof.* (1)  $\implies$  (2): This follows from an easy approximation argument, using the uniform density of  $\mathcal{F}_n$  in the space of continuous functions from the interval  $[-n, n]$  to itself.

(2)  $\implies$  (3): Use the following standard trick. If  $\varphi$  is an existential sentence then there is a large enough  $N \in \mathbb{N}$  such that  $N - \varphi$  is semantically equivalent to a universal sentence. Moreover  $(N - \varphi)^M = N - \varphi^M$  for every  $\mathcal{L}$ -structure  $M$ .

(3)  $\implies$  (4): This follows easily from the definition of the Gromov–Hausdorff distance.

(4)  $\implies$  (5): Fix countable subsets  $M_0$  and  $N_0$  of  $M$  and  $N$  such that  $M_0 \cap D_k(M)$  is dense in  $D_k(M)$  and  $N_0 \cap D_k(N)$  is dense in  $D_k(N)$  for every  $k \in \mathbb{N}$ . Then for every  $k \in \mathbb{N}$  there is a  $(D_k, \mathcal{L}_k, 2^{-k})$ -approximate

isomorphism  $\psi$  from  $M$  to  $N$ . This allows one to define functions  $f_k : M \rightarrow N$  such that  $f_k[D_k(M)] \subset D_k(N)$  and for every  $n$ -ary relation  $B$  in  $\mathcal{L}_k$ , every  $\bar{x} \in (M_0 \cap D_k(M))^n$

$$\psi^B(\bar{x}, f_k(\bar{x})) < 2^{-k}$$

and for every  $y \in N_0 \cap D_k(M)$  there is  $z \in M_0 \cap D_k(M)$  such that

$$d(f_k(z), y) < 2^{-k}.$$

By compactness of  $D_k(N)$ , after passing to a subsequence we can assume that the sequence  $(f_k(x))_{k \in \mathbb{N}}$  converges in  $N$  for every  $x \in M_0$ . Defining  $f_\infty(x)$  to be the limit of the sequence  $(f_k(x))_{k \in \mathbb{N}}$  defines an isometry  $f_\infty$  from  $M_0$  onto a dense subset of  $N_0$  that preserves all the relations  $\mathcal{L}$ . The unique isometric extension of  $f_\infty$  to the whole  $M$  defines an isomorphism from  $M$  onto  $N$ .

5)  $\implies$  (6)  $\implies$  (1) These are clear by definition.  $\square$

Proposition 5.3 can be regarded as the metric analogue of the classical fact that a finite discrete structure is classified by its first order theory.

**Corollary 5.4.** *The isomorphism relation for proper  $\mathcal{L}$ -structures is smooth.*

*Proof.* It can be easily shown by induction on the complexity of a formula  $\varphi(x_1, \dots, x_n)$  that the interpretation function

$$(M, a_1, \dots, a_n) \mapsto \varphi^M(a_1, \dots, a_n)$$

is a Borel function from  $\text{Mod}(\mathcal{L}) \times \mathbb{N}^n$  to  $\mathbb{R}$ . This has been shown in the particular case of  $C^*$ -algebras in [21, Proposition 5.1]. Fix an enumeration  $(\varphi_n)_{n \in \mathbb{N}}$  of all restricted universal sentences. By Proposition 5.3 the function

$$M \mapsto (\varphi_n^M)_{n \in \mathbb{N}}$$

from  $\text{Mod}_p(\mathcal{L})$  to  $\mathbb{R}^{\mathbb{N}}$  is a Borel reduction from the relation of isomorphism of proper  $\mathcal{L}$ -structures to equality of sequences of real numbers.  $\square$

An alternative way to obtain Corollary 5.4 is to define the complete separable metric space  $X_{\mathcal{L}}$  of isomorphism classes of proper  $\mathcal{L}$ -structures endowed with the Gromov–Hausdorff metric  $d_{GH}$ . The equivalence of (4) and (5) in Proposition 5.3 shows that  $(X_{\mathcal{L}}, d_{GH})$  is indeed a metric space. After coding any function  $f$  by the relation expressing the distance from the graph of  $f$ , we can assume that  $\mathcal{L}$  contains only relation symbols. Thus separability follows by considering the countable dense collection of finite  $\mathcal{L}$ -structures with rational-valued relations. Finally completeness follows from the standard argument allowing to build the Gromov–Hausdorff limit of a Cauchy sequence; see for example [37, Proposition 43]. The proof is concluded observing that the function that assigns to a code for an  $\mathcal{L}$ -structure  $M \in \text{Mod}_p(\mathcal{L})$  its isomorphism class  $[M] \in X_{\mathcal{L}}$  is a Borel reduction from isomorphism to equality.

In view of the Choi–Effros abstract characterization [36, Theorem 13.1], operator systems can be described as structures in a suitable language  $\mathcal{L}_{OSy}$

in [27, Appendix B]; see also [19, Section 3.3]. In this case the domains of quantifications are norm balls of matrix amplifications. Therefore the following is an immediate consequence of Proposition 5.3.

**Corollary 5.5.** *Suppose that  $X$  and  $Y$  are finitely generated operator systems. Then  $X$  and  $Y$  are complete order isomorphic if and only if  $\varphi^X = \varphi^Y$  for every universal  $\mathcal{L}_{OSy}$ -sentence. In particular the complete order isomorphism for finitely generated operator systems is smooth.*

The same result equally applies to finitely generated operator spaces. These can also be regarded as structures in a language  $\mathcal{L}_{OSp}$  [27, Appendix B], using Ruan's abstract characterization [36, Theorem 13.4].

**Corollary 5.6.** *Suppose that  $X$  and  $Y$  are finitely generated operator spaces. Then  $X$  and  $Y$  are completely isometric if and only if  $\varphi^X = \varphi^Y$  for every universal  $\mathcal{L}_{OSy}$ -sentence. In particular the complete isometry relation for finitely generated operator spaces is smooth.*

Finally one can consider finitely generated *unital* operator spaces. The abstract characterization provided by [9, Theorem 1.1] shows that unital operator spaces can be regarded as  $\mathcal{L}_{uOSp}$ -structures in a suitable language  $\mathcal{L}_{uOSp}$ .

**Corollary 5.7.** *Suppose that  $X$  and  $Y$  are finitely generated operator spaces. Then  $X$  and  $Y$  are unittally completely isometric if and only if  $\varphi^X = \varphi^Y$  for every universal  $\mathcal{L}_{uOSy}$ -sentence. In particular the unital complete isometry relation for finitely generated unital operator spaces is smooth.*

## APPENDIX A. EQUIVALENCE OF PARAMETRIZATIONS OF OPERATOR SYSTEMS

In this Appendix we show that the parametrizations of operator systems  $\Gamma$ ,  $\Xi$ , and  $\widehat{\Xi}$  are equivalent. Furthermore we show that  $\Gamma_N$  and  $\widehat{\Gamma}_N$  provide weakly equivalent parametrizations of  $N$ -dimensional operator systems.

The argument of the following lemma is analogous to the one of [22, Lemma 2.4]. The full proof is presented for the convenience of the reader.

**Lemma A.1.** *Suppose that  $X$  is a standard Borel space, and  $Y$  is any of the spaces  $\Gamma$ ,  $\Xi$  or  $\widehat{\Xi}$ . Let  $f$  be a Borel function from  $X$  to  $Y$ . Then there is a Borel injection  $\tilde{f}$  from  $X$  to  $Y$  such that  $\mathcal{OSy}(\tilde{f}(x)) \cong \mathcal{OSy}(f(x))$  for every  $x \in X$ .*

*Proof.* Consider the case where  $Y = \Gamma$ . Without loss of generality we can assume that  $X$  is the standard Borel space of infinite subsets of  $\mathbb{N}$  regarded as a subset of  $2^{\mathbb{N}}$ . Denote by  $I$  the identity operator on  $H$ . Define the function  $\tilde{f}: X \rightarrow \Gamma$  by setting

$$\tilde{f}(A)_k = \begin{cases} nI & \text{if } k = 2^n \text{ for some } n \in A, \\ f(A)_n & \text{if } k = 3^n \text{ for some } n \in \mathbb{N}, \\ 0 & \text{otherwise.} \end{cases}$$



Then the function  $\tilde{f}$  is a Borel injection and  $\mathcal{OSy}(\tilde{f}(A)) \cong \mathcal{OSy}(f(A))$  for every  $A \in X$ .

Consider now the case when  $Y = \widehat{\Xi}$ . Without loss of generality we can assume that  $X$  is the standard Borel space of infinite subsets of the set of *even* natural numbers. Observe that the group  $S_\infty$  of permutations of  $\mathbb{N}$  has a natural action on  $\widehat{\Xi}$ . Explicitly if  $(f, g, h, (C_\xi)_{n \in \mathbb{N}}, e_\xi) \in \widehat{\Xi}$  and  $\sigma \in S_\infty$  then  $\sigma \cdot \xi$  is the element of  $\widehat{\Xi}$  obtained replacing  $f$  with the function  $(n, m) \mapsto \sigma(f(\sigma^{-1}(n), \sigma^{-1}(m)))$  and similarly with the other entries of  $\xi$ . It is clear that  $\mathcal{OSy}(\xi) \cong \mathcal{OSy}(\sigma \cdot \xi)$  for any  $\xi \in \widehat{\Xi}$  and  $\sigma \in S_\infty$ . Given  $\xi \in \widehat{\Xi}$  and  $A \in X$  one can find in a Borel way a permutation  $\sigma_{\xi, A}$  of  $\mathbb{N}$  such that

$$A = \{2^n \cdot \sigma_{\xi, A} 1 : n \in \mathbb{N}\}.$$

We can then define the Borel injection  $\tilde{f} : X \rightarrow \widehat{\Xi}$  by

$$\tilde{f}(A) = \sigma_{f(A), A} \cdot f(A).$$

Finally suppose that  $Y = \Xi$ . We can assume without loss of generality that  $X$  is the space of positive real numbers. Fix  $x \in X$  and define  $\delta = f(x)$ . Consider the least  $n_0 \in \mathbb{N}$  such that  $\delta_1(X_{n_0}) \neq 0$ . Let  $X$  be an operator system and  $\gamma$  be a dense sequence in  $X$  such that, for every  $[q_{ij}] \in M_n(\mathcal{V})$ ,

$$\delta_n([q_{ij}]) = \|[q_{ij}(\gamma)]\|_{M_n(A)}.$$

Let  $\tilde{\gamma}$  to be the sequence in  $X$  defined by

$$\tilde{\gamma}_n = \begin{cases} \frac{x}{\|\gamma_i\|_A} \gamma_i & \text{if } i = n_0, \\ \gamma_i & \text{otherwise.} \end{cases}$$

Define

$$\tilde{f}(x)_n([q_{ij}]) = \|[q_{ij}(\tilde{\gamma})]\|_{M_n(A)}.$$

Observe that  $\tilde{f}$  is well defined and injective. Note also that  $\tilde{f}(x)_n = \delta'$  if and only if there are  $\gamma, \gamma' \in \Gamma$  and  $\delta \in \Xi$  such that  $\|p(\gamma)\| = \delta(p)$  and for every  $p \in \mathcal{V}$ ,  $n \in \mathbb{N}$ , and  $q_{ij} \in M_n(\mathcal{V})$ , and  $\delta'([q'_{ij}]) = \delta([q_{ij}])$ . Here  $q'_{ij}$  is the polynomial obtained from  $q_{ij}$  by replacing any occurrence of  $X_{n_0}$  with  $(\delta_1(X_{n_0})/x) X_{n_0}$ , while  $n_0$  is defined as above to be the least natural number such that  $\delta_1(X_{n_0})$  is nonzero. This observation gives an analytic definition for the function  $\tilde{f}$ . The classical principle that a function with analytic graph is Borel [34, Theorem 14.12] concludes the proof.  $\square$

**Proposition A.2.** *The parametrizations of operator systems  $\Gamma$ ,  $\Xi$ , and  $\widehat{\Xi}$  are equivalent.*

*Proof.* In view of Lemma A.1 it is enough to show that the parametrizations  $\Gamma$ ,  $\Xi$ , and  $\widehat{\Xi}$  are weakly equivalent. Suppose that  $\gamma \in \Gamma$ . Define the matrix

ordered  $\mathbb{Q}(i)$ -\*-vector space structure  $\xi_\gamma$  on  $\mathbb{N}$

$$\begin{aligned} n +_{\xi_\gamma} m = k &\iff \mathfrak{p}_n(\gamma) + \mathfrak{p}_m(\gamma) = \mathfrak{p}_k(\gamma), \\ q \cdot_{\xi_\gamma} n = k &\iff q\mathfrak{p}_n(\gamma) = \mathfrak{p}_k(\gamma), \\ n^{*\xi_\gamma} = k &\iff \mathfrak{p}_n(\gamma)^* = \mathfrak{p}_k(\gamma). \end{aligned}$$

The positive cones are defined by

$$[m_{ij}] \in C_{\xi,n} \iff [\mathfrak{p}_{m_{ij}}(\gamma)] \geq 0.$$

Finally  $e_{\xi_\gamma} = 1$  is an Archimedean matrix order unit. This defines a Borel function  $\gamma \mapsto \xi_\gamma$  from  $\Gamma$  to  $\widehat{\Xi}$  such that  $\mathcal{OSy}(\gamma) \cong \mathcal{OSy}(\xi_\gamma)$ .

Now suppose that  $\xi \in \widehat{\Xi}$ . If  $p \in \mathcal{V}$  define  $p^\xi$  to be the element of  $\mathbb{N}$  obtained by evaluating  $p$  in the  $\mathbb{Q}(i)$ -\*-vector space structure on  $\mathbb{N}$  defined by  $\xi$  after by replacing  $X_i$  with  $i$  and replacing the constant  $c$  by  $c \cdot_\xi e_\xi$ . Denote by  $rI_n^\xi$  the  $n \times n$  matrix with positive integer coefficients having  $r \cdot_\xi e_\xi$  in the diagonal entries and  $0 \cdot_\xi e_\xi$  elsewhere. Define  $\delta_\xi \in \Xi$  by setting, for  $P \in M_n(\mathcal{V})$  and  $r \in \mathbb{Q}_+$ ,

$$\delta_{\xi,n}(P) < r \iff \begin{bmatrix} rI_n^\xi & P \\ P^* & rI_n^\xi \end{bmatrix} \in C_{\xi,2n}.$$

By [36, Proposition 13.3] this defines a Borel map  $\xi \mapsto \delta_\xi$  from  $\widehat{\Xi}$  to  $\Xi$  such that  $\mathcal{OSy}(\xi) \cong \mathcal{OSy}(\delta_\xi)$ .

To conclude the proof it is now enough to describe a Borel function  $\delta \mapsto \gamma_\delta$  from  $\Xi$  to  $\Gamma$  such that  $\mathcal{OSy}(\gamma_\delta) \cong \mathcal{OSy}(\delta)$ . For  $\delta \in \Xi$  and  $k \in \mathbb{N}$  define  $P_k(\delta)$  the set of  $\phi \in M_k(\mathbb{C})^\mathcal{V}$  such that

- $\|[\phi(p_{ij})]\|_{M_{kn}(\mathbb{C})} \leq \delta_n([p_{ij}])$  for every  $n \in \mathbb{N}$  and  $[p_{ij}] \in M_n(\mathcal{V})$ , and
- $\phi(\mathfrak{p}_1)$  is the identity matrix.

(Recall that  $\mathfrak{p}_1$  is the constant polynomial 1.) Then  $P_k(\delta)$  is a compact subset of  $M_k(\mathbb{C})^\mathcal{V}$  with the product topology. Moreover the relation

$$\{(\delta, P_k(\delta)) \in \Xi \times M_k(\mathbb{C})^\mathcal{V} : \phi \in P_k(\delta)\}$$

is Borel. It follows from [34, Theorem 28.8] that the function

$$\delta \mapsto P_k(\delta)$$

is a Borel function into the Polish space  $K(M_k(\mathbb{C})^\mathcal{V})$  of compact subsets of  $M_k(\mathbb{C})^\mathcal{V}$ . Consider the Borel set  $A_n$  of tuples  $(\delta, \varepsilon, [m_{ij}], k, \phi)$  where

- $\delta \in \Xi$ ,
- $\varepsilon \in \mathbb{Q}_+$ ,
- $[p_{ij}] \in M_n(\mathcal{V})$ ,
- $k \in \{1, 2, \dots, n\}$ , and
- $\phi \in P_k(\delta)$  is such that  $\|[\phi(p_{ij})]\|_{kn} \geq \delta_n([p_{ij}]) - \varepsilon$ .

The proof of the Choi-Effros abstract characterization of operator systems [36, page 179] shows that for every  $(\delta, \varepsilon, [p_{ij}]) \in \Xi \times \mathbb{Q}_+ \times M_n(\mathcal{V})$

the corresponding section of  $A_n$  is (compact and) nonempty. Therefore by [34, Theorem 28.8] there is a Borel function

$$(\delta, \varepsilon, [p_{ij}]) \mapsto \phi_{\delta, \varepsilon, [p_{ij}]} \in P_{k_{\delta, \varepsilon, [p_{ij}]}}(\delta)$$

such that  $\|[\phi(p_{ij})]\|_{k_{\delta, \varepsilon, [p_{ij}]}n} \geq \delta_n([p_{ij}]) - \varepsilon$ . Denote by  $\mathcal{M}$  the set of  $n \times n$  matrices  $[p_{ij}]$  with entries in  $\mathcal{V}$  where  $n$  varies in  $\mathbb{N}$ . Denote by  $H$  the separable Hilbert space with orthonormal basis  $(e_{[p_{ij}], \varepsilon, \alpha})$  indexed by  $\mathcal{M} \times \mathbb{Q}_+ \times \mathbb{N}$ . For every  $k \in \mathbb{N}$  denote by  $(b_{n, \alpha})_{\alpha \leq n}$  the canonical basis of  $\mathbb{C}^k$ . For  $\delta \in \widehat{\Xi}$  and  $n \in \mathbb{N}$  denote by  $\gamma_{\delta, n}$  the element of  $B(H)$  defined by setting

$$\left\langle \gamma_{\delta, n} e_{[p_{ij}], \varepsilon, \alpha}, e_{[q_{ij}], \varepsilon, \beta} \right\rangle = \left\langle \phi_{\delta, \varepsilon, [p_{ij}]}(X_n) e_{k_{\delta, \varepsilon, [p_{ij}]}, \alpha}, b_{k_{\delta, \varepsilon, [p_{ij}]}, \beta} \right\rangle$$

if  $p_{ij} = q_{ij}$  and  $\alpha, \beta \leq k_{\delta, \varepsilon, [p_{ij}]}$ , and zero otherwise. The construction ensures that the map  $p \mapsto p(\gamma)$  extends to a complete isometry from the operator system  $\mathcal{OSy}(\delta)$  coded by  $\delta$  and the operator system  $\mathcal{OSy}(\gamma_\delta)$  coded by the sequence  $\gamma_\delta = (\gamma_{\delta, n})_{n \in \mathbb{N}}$ . Observing that the function  $\delta \mapsto \gamma_\delta$  is Borel concludes the proof.  $\square$

We will now verify that the sets  $\Gamma_N$  and  $\widehat{\Gamma}_N$  provide weakly equivalent parametrizations of the  $N$ -dimensional operator systems. Recall that  $\Gamma_N$  is the set of  $\gamma \in \Gamma$  such that  $\mathcal{OS}(\gamma)$  has dimension  $N$ . Similarly  $\widehat{\Gamma}_N$  is the set of linearly independent tuples  $(x_1, \dots, x_N)$  in  $B(H)$  such that  $\text{span}\{x_1, \dots, x_N\}$  is an operator system. Denote as before by  $\mathcal{V}_{\mathbb{C}}$  the complex  $*$ -vector space of noncommutative  $*$ -polynomials with coefficients from  $\mathbb{C}$ . Endow  $\mathcal{V}_{\mathbb{C}}$  with the norm

$$\left\| \sum_{i \leq n} a_i X_i + \sum_{i \leq n} b_i X_i^* + c \right\| = \sum_{i \leq n} |a_i| + \sum_{i \leq n} |b_i| + |c|.$$

Let us show that the set  $\Gamma_{\leq N}$  of  $\gamma \in \Gamma$  such that  $\mathcal{OSy}(\gamma)$  has dimension at most  $N$  is Borel. To this purpose it is enough to show that it is both analytic and coanalytic [34, Theorem 14.1]. Observe that on one hand  $\gamma \in \Gamma_{\leq N}$  if and only if there are  $p_1, \dots, p_N \in \mathcal{V}_{\mathbb{C}}$  such that for every  $n \in \mathbb{N}$  and  $q \in \mathcal{V}$  there is  $r \in \mathcal{V}$  such that

$$\|r(p_1(\gamma), \dots, p_N(\gamma)) - q(\gamma)\| < \frac{1}{n}.$$

On the other hand  $\gamma \in \Gamma_{\leq N}$  if and only if for every  $p_1, \dots, p_{N+1} \in \mathcal{V}_{\mathbb{C}}$  there are  $\lambda_1, \dots, \lambda_{N+1} \in \mathbb{Q}(i)$  such that  $\lambda_i \neq 0$  for some  $i \in \{1, 2, \dots, N+1\}$  and

$$\lambda_1 p_1 + \dots + \lambda_{N+1} p_{N+1} = 0.$$

This shows that  $\Gamma_{\leq N}$  is both analytic and coanalytic, and hence Borel.

**Lemma A.3.** *There are Borel functions  $\gamma \mapsto b_i^\gamma$  from  $\Gamma_N$  to  $\mathcal{V}$  for  $i \leq N$  such that  $\{b_1^\gamma(\gamma), \dots, b_N^\gamma(\gamma)\}$  is a basis for  $\mathcal{OSy}(\gamma)$ .*

*Proof.* We show by induction on  $k \leq N$  that there are Borel functions  $\gamma \mapsto b_i^\gamma$  from  $\Gamma_N$  to  $\mathcal{V}$  for  $i \leq k$  such that  $\{b_1^\gamma(\gamma), \dots, b_N^\gamma(\gamma)\}$  is a linearly independent set. For  $k = 1$  this is immediate. Suppose that  $b_1^\gamma, \dots, b_k^\gamma$  have been defined for  $k < N$ . Define the relation  $A$  to be the set of pairs  $(\gamma, p) \in \Gamma_N \times \mathcal{V}$  such that  $p(\gamma) \notin \text{span}\{b_1^\gamma(\gamma), \dots, b_k^\gamma(\gamma)\}$ . Since for every  $\gamma \in \Gamma_N$  the corresponding section  $A_\gamma$  is nonempty, there is a Borel function  $b_{k+1}^\gamma : \Gamma_N \rightarrow \mathcal{V}$  such that  $(\gamma, b_{k+1}^\gamma(\gamma)) \in A$ .  $\square$

Denote in the following by  $b_1^\gamma, \dots, b_N^\gamma$  the functions defined in Lemma A.3. Observe that the maps  $\gamma \mapsto b_k^\gamma$  from  $\Gamma$  to  $B(H)$  are Borel. Denote as before by  $\mathcal{W}_N$  the set of polynomials of degree 1 in the noncommutative variables  $X_1, \dots, X_N$  and with coefficients from  $\mathbb{Q}(i)$ . Similarly denote by  $\mathcal{V}_N$  the set of polynomials of degree 1 in the noncommutative variables  $X_1, \dots, X_N, X_1^*, \dots, X_N^*$  and with coefficients from  $\mathbb{Q}(i)$ .

**Lemma A.4.**  $\widehat{\Gamma}_N$  is a Borel subset of  $B(H)^N$ .

*Proof.* Observe that  $\bar{x} \in \widehat{\Gamma}_N$  if and only if

- for every  $\varepsilon \in \mathbb{Q}_+$  there is  $p \in \mathcal{V}_N$  such that  $\|p(x) - I\| < \varepsilon$ ,
- for every  $p \in \mathcal{V}_N$  and  $\varepsilon \in \mathbb{Q}_+$  there is  $q \in \mathcal{W}_N$  such that

$$\|p(x) - q(x)\| < \varepsilon,$$

and

- for every  $k < N$  there is  $\varepsilon \in \mathbb{Q}_+$  such that for every  $q \in \mathcal{W}_k$

$$\|q(x_1, \dots, x_k) - x_{k+1}\| \geq \varepsilon. \quad \square$$

The operator system associated with  $(x_1, \dots, x_N) \in \widehat{\Gamma}_N$  is the span of  $\{x_1, \dots, x_N\}$ . Lemma A.4 shows that  $\widehat{\Gamma}_N$  is a standard Borel parametrization of  $N$ -dimensional operator systems. By Lemma A.3 such a parametrization is weakly equivalent to the parametrization  $\Gamma_N$ .

## APPENDIX B. EQUIVALENCE OF PARAMETRIZATIONS OF OPERATOR SPACES

In this appendix we show that the parametrizations for operator spaces  $\Xi$ ,  $\widehat{\Xi}$ , and  $\Gamma$  are equivalent. The proof of the following lemma is entirely analogous to the proof of Lemma A.1 and [22, Lemma 2.4].

**Lemma B.1.** *Suppose that  $X$  is a standard Borel space, and  $Y$  is any of the space  $\Gamma$ ,  $\Xi$ , and  $\widehat{\Xi}$ . If  $f$  is a Borel function from  $X$  to  $Y$ , then there is a Borel injection  $\tilde{f}$  from  $X$  to  $Y$  such that  $\mathcal{O}Sp(f(x)) \cong \mathcal{O}Sp(\tilde{f}(x))$  for every  $x \in X$ .*

**Proposition B.2.** *The parametrizations of operator spaces  $\Gamma$ ,  $\Xi$ , and  $\widehat{\Xi}$  are equivalent.*

*Proof.* In view of Lemma B.1 it is enough to show that  $\Gamma$ ,  $\Xi$ , and  $\widehat{\Xi}$  are weakly equivalent. Isomorphism-preserving Borel functions from  $\Gamma$  to  $\widehat{\Xi}$  and from  $\widehat{\Xi}$  to  $\Xi$  can be easily defined as in Proposition A.2. Hence we focus here on constructing an isomorphism-preserving Borel function from  $\Xi$  to  $\Gamma$ . Observe that we can identify  $\mathcal{V}$  with the  $\mathbb{Q}(i)$ -\*-vector space  $\mathbb{Q}(i) \oplus \mathbb{Q}(i) \oplus \mathcal{W} \oplus \overline{\mathcal{W}}$ , where  $\overline{\mathcal{W}}$  denote the complex conjugate of the  $\mathbb{Q}(i)$ -vector spaces  $\mathcal{W}$ . For convenience we represent an element of  $\mathcal{V}$  as a matrix

$$\begin{bmatrix} \lambda & p \\ q^* & \mu \end{bmatrix}$$

where  $\lambda, \mu \in \mathbb{Q}(i)$  and  $p, q \in \mathcal{W}$ . Similarly an  $n \times n$  matrix  $V$  of elements of  $\mathcal{V}$  can be regarded, after a canonical shuffle, as

$$\begin{bmatrix} P & X \\ Y^* & Q \end{bmatrix}$$

where  $P, Q \in M_n(\mathbb{Q}(i))$  and  $X, Y \in M_n(\mathcal{W})$ . We will adopt these identifications throughout the rest of the proof. Suppose that  $\delta \in \Xi$ . Define  $C_n$  to be the set of

$$\begin{bmatrix} P & X \\ X^* & Q \end{bmatrix}$$

such that  $P, Q \in M_n(\mathbb{Q}(i))$  are positive, and

$$\left\| (P + \varepsilon I_n)^{-1} X (Q + \varepsilon I_n)^{-1} \right\| \leq 1$$

for every  $\varepsilon \in \mathbb{Q}_+$ . Define then for  $V \in M_n(\mathcal{V})$

$$\widehat{\delta}_n(V) < r \iff \begin{bmatrix} I_n & V \\ V^* & I_n \end{bmatrix} \in C_{2n}.$$

The proof of the abstract characterization of operator spaces due to Ruan [36, Theorem 13.4] shows that

$$(\widehat{\delta}_n)_{n \in \mathbb{N}} \in \prod_{n \in \mathbb{N}} \mathbb{R}^{M_n(V)}$$

is a code for an operator system in the parametrization  $\Xi$  for operator systems. Moreover the function

$$q \mapsto \begin{pmatrix} 0 & q \\ 0 & 0 \end{pmatrix}$$

induces a complete isometry from the operator space coded by  $\delta$  into the operator system coded by  $\widehat{\delta}$ . The proof of Proposition A.2 allows one to assign in a Borel way to  $\widehat{\delta}$  a sequence  $\gamma_\delta$  in  $B(H)$  such that the function  $p \mapsto p(\gamma_\delta)$  induces a complete isometry from the operator system coded by  $\widehat{\delta}$  onto the operator system generated by  $\gamma_\delta$ . Therefore the function  $q \mapsto q(\gamma_\delta)$  induces a complete isometry from the operator space coded by  $\delta$  onto the operator space generated by  $\gamma_\delta$ . The proof is concluded by observing that the construction above shows that the assignment  $\delta \mapsto \gamma_\delta$  is Borel.  $\square$

## REFERENCES

- [1] W. Arveson. Subalgebras of  $C^*$ -algebras. *Acta Mathematica*, 123(1):141–224, Dec. 1969.
- [2] W. Arveson. Subalgebras of  $C^*$ -algebras II. *Acta Mathematica*, 128(1):271–308, July 1972.
- [3] W. Arveson. The noncommutative Choquet boundary. *Journal of the American Mathematical Society*, 21(4):1065–1084, 2008.
- [4] W. Arveson. The noncommutative Choquet boundary III: operator systems in matrix algebras. *Mathematica Scandinavica*, 106(2):196–210, 2010.
- [5] I. Ben Yaacov. Fraïssé limits of metric structures. *Journal of Symbolic Logic*. to appear.
- [6] I. Ben Yaacov, A. Berenstein, C. W. Henson, and A. Usvyatsov. Model theory for metric structures. In *Model theory with applications to algebra and analysis. Vol. 2*, volume 350 of *London Mathematical Society Lecture Note Series*, pages 315–427. Cambridge University Press, 2008.
- [7] I. Ben Yaacov, A. Nies, and T. Tsankov. A Lopez-Escobar theorem for continuous logic. *arXiv:1407.7102*, 2014.
- [8] D. P. Blecher and C. Le Merdy. *Operator algebras and their modules—an operator space approach*, volume 30 of *London Mathematical Society Monographs. New Series*. Oxford University Press, Oxford, 2004. Oxford Science Publications.
- [9] D. P. Blecher and M. Neal. Metric characterizations of isometries and of unital operator spaces and systems. *Proceedings of the American Mathematical Society*, 139(3):985–998, 2011.
- [10] D. P. Blecher and M. Neal. Metric characterizations II. *Illinois Journal of Mathematics*, 57(1):25–41, 2013.
- [11] B. G. Bodmann, D. W. Kribs, and V. I. Paulsen. Decoherence-insensitive quantum communication by optimal  $C^*$ -encoding. *IEEE Transactions on Information Theory*, 53(12):4738–4749, Dec. 2007.
- [12] B. G. Bodmann and V. I. Paulsen. Frame paths and error bounds for sigma-delta quantization. *Applied and Computational Harmonic Analysis*, 22(2):176–197, Mar. 2007.
- [13] B. G. Bodmann, V. I. Paulsen, and S. A. Abdalbaki. Smooth frame-path termination for higher order sigma-delta quantization. *Journal of Fourier Analysis and Applications*, 13(3):285–307, June 2007.
- [14] B. Bollobás. *Linear analysis*. Cambridge University Press, Cambridge, second edition, 1999. An introductory course.
- [15] M.-D. Choi and E. G. Effros. Injectivity and operator spaces. *Journal of Functional Analysis*, 24(2):156–209, Feb. 1977.
- [16] K. R. Davidson and M. Kennedy. The Choquet boundary of an operator system. *Duke J. Math.*. to appear.
- [17] M. A. Dritschel and S. A. McCullough. Boundary representations for families of representations of operator algebras and spaces. *Journal of Operator Theory*, 53(1):159–167, 2005.
- [18] G. A. Elliott. On the classification of inductive limits of sequences of semisimple finite-dimensional algebras. *Journal of Algebra*, 38(1):29–44, Jan. 1976.
- [19] G. A. Elliott, I. Farah, V. Paulsen, C. Rosendal, A. S. Toms, and A. Törnquist. The isomorphism relation for separable  $C^*$ -algebras. *Mathematical Research Letters*, 20(6):1071–1080, 2013.
- [20] I. Farah, B. Hart, and D. Sherman. Model theory of operator algebras II: Model theory. *Israel Journal of Mathematics*, 201 (2014), no. 1, 477505.
- [21] I. Farah, A. Toms, and A. Törnquist. The descriptive set theory of  $C^*$ -algebra invariants. *International Mathematics Research Notices*, page rns206, Sept. 2012.

- [22] I. Farah, A. S. Toms, and A. Törnquist. Turbulence, orbit equivalence, and the classification of nuclear  $C^*$ -algebras. *Journal für die reine und angewandte Mathematik*, 688:101–146, Mar. 2014.
- [23] D. Farenick, A. S. Kavruk, V. I. Paulsen, and I. G. Todorov. Characterisations of the weak expectation property. *arXiv:1307.1055*, July 2013.
- [24] D. Farenick and V. I. Paulsen. Operator system quotients of matrix algebras and their tensor products. *Mathematica Scandinavica*, 111(2):210–243, 2012.
- [25] V. Ferenczi, A. Louveau, and C. Rosendal. The complexity of classifying separable Banach spaces up to isomorphism. *Journal of the London Mathematical Society*, 79(2):323–345, Apr. 2009.
- [26] J. G. Glimm. On a certain class of operator algebras. *Transactions of the American Mathematical Society*, 95(2):318–340, 1960.
- [27] I. Goldbring and T. Sinclair. On Kirchberg’s embedding problem. *arXiv:1404.1861*, Apr. 2014. *arXiv: 1404.1861*.
- [28] M. Hamana. Injective envelopes of  $C^*$ -algebras. *Journal of the Mathematical Society of Japan*, 31(1):181–197, Jan. 1979.
- [29] M. Hamana. Injective envelopes of operator systems. *Publications of the Research Institute for Mathematical Sciences*, 15(3):773–785, 1979.
- [30] G. Hjorth. *Classification and Orbit Equivalence Relations*, volume 75 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 2000.
- [31] N. Johnston, D. W. Kribs, and V. I. Paulsen. Computing stabilized norms for quantum operations via the theory of completely bounded maps. *Quantum Information & Computation*, 9(1-2):16–35, 2009.
- [32] N. Johnston, D. W. Kribs, V. I. Paulsen, and R. Pereira. Minimal and maximal operator spaces and operator systems in entanglement theory. *Journal of Functional Analysis*, 260(8):2407–2423, 2011.
- [33] A. S. Kavruk. Nuclearity related properties in operator systems. *J. Operator Theory*, 71 (2014), no. 1, 95156.
- [34] A. S. Kechris. *Classical descriptive set theory*, volume 156 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1995.
- [35] K. Latrach, J. M. Paoli, and P. Simonnet. Some facts from descriptive set theory concerning essential spectra and applications. *Studia Mathematica*, 171(3):207–225, 2005.
- [36] V. I. Paulsen. *Completely bounded maps and operator algebras*, volume 78 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 2002.
- [37] P. Petersen. *Riemannian geometry*, volume 171 of *Graduate Texts in Mathematics*. Springer, New York, second edition, 2006.
- [38] G. Pisier. *Introduction to operator space theory*, volume 294 of *London Mathematical Society Lecture Note Series*. Cambridge University Press, Cambridge, 2003.
- [39] Z.-j. Ruan. Subspaces of  $C^*$ -algebras. *Journal of Functional Analysis*, 76(1):217–230, Jan. 1988.
- [40] M. Sabok. Completeness of the isomorphism problem for separable  $C^*$ -algebras. *arXiv:1306.1049*, June 2013.
- [41] R. Sasyk and A. Törnquist. The classification problem for von Neumann factors. *Journal of Functional Analysis*, 256(8):2710–2724, Apr. 2009.
- [42] J. Zielinski. The complexity of the homeomorphism relation between compact metric spaces. *arXiv:1409.5523*, Sept. 2014.

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